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REFORMED MATHEMATICAL TEACHING[1]

By R. C. FAWDRY, M.A., B.Sc.
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The agitation for a reform in the teaching of mathematics began as a revolt against the authority of Euclid whose dead hand fifty years ago still held a close grip upon the teaching of geometry in this country. England was his last stronghold. He had been supplanted in France during the latter half of the eighteenth century by Lacroix, Legendre, and d'Alembert, who introduced practical work into their geometry, accepted proofs which ignored the case of irrationals, and did not despise intuition as a means of acquiring geometrical knowledge. America followed the lead of France, and England at that time was the only country where Euclid was the only text-book.

Venerable though the figure of Euclid may have been, his reign in schools was much shorter than is generally supposed; for there is no evidence that the study of Euclid began in public schools before the nineteenth century, and it did not become general in secondary schools until the middle of that century.

We have Lord Redesdale's authority for saying that at Eton until 1851 mathematics formed no part of the school curriculum. He tells us that when mathematics were introduced, Mr. Hawtrey, the master appointed (who, by the way, was not allowed to wear cap and gown), had to examine the boys to divide them into classes. Naturally every one tried to make as bad a show as possible to get into a low set. "When my form came up," he says, "question after question did the unhappy man put—no answer." At last in despair he said, "Can no one here tell me what twice two makes?" After a pause,

"Yes, sir, please sir, I can."

"Well, what is it?"

"Five, sir, please sir."

Even in England Euclid was not without his critics. de

[1] Reprinted from The Journal of Education and School World, September, 1923.

Morgan, a voice crying in the wilderness fifty years before his message was appreciated, advocated a preliminary course of measurements, adopted arithmetical methods for the treatment of proportion and in many places substituted Legendre for Euclid.

In 1869 at the meeting of the British Association, Sylvester said he would like to see Euclid buried deeper than e'er plummet sounded out of the schoolboy's reach, and in 1871 the Association for the Improvement of Geometrical Teaching, now the Mathematical Association, was formed, and Euclid was doomed.

The agitation for the reform of geometrical teaching was followed by a demand from the technical colleges that the engineering students, ever increasing in number, should come to them equipped with a wider knowledge of mathematics and the power of applying this knowledge to practical problems. This demand was voiced by Professor Perry, who took so prominent a part in the matter that the reforms, both in geometry and in other branches of mathematics, are known here and abroad as the Perry Movement.

At the British Association meeting in 1905, Professor Perry sketched a scheme of education for a boy who at the age of fourteen had learnt elementary trigonometry—could use logarithms, knows what is meant by speed—can differentiate and integrate x^n, has had many applications of the use of the calculus, has a *fairly* clear notion of the various forms of energy—knows about the law of work and friction—has experimented on the efficiency of machines, can calculate K.E., and now measures the strength and stiffness of wires and beams. Most schoolmasters have yet to meet this boy of fourteen—they would be well content if this standard were reached by the average boy of sixteen. Canon Wilson, who took a leading part in the foundation of the A.I.G.T. fifty-one years ago, is still alive and in harness; he has seen the reforms he advocated in geometry universally adopted, but even if Professor Perry had, like Canon Wilson, lived to the ripe old age of eighty he would still be complaining that the schoolmaster had not yet produced this prodigy who will remain, I fear, a mere figment of the professor's imagination.

Broadly speaking, Professor Perry, when he came down to solid earth, wished boys to leave school with a sound knowledge

of mechanics; that this might be possible it was necessary to shorten and simplify the earlier stages of the mathematical curriculum and the doing of this incidentally led to a valuable reconsideration of the purport of mathematics as a fundamental part of a general education.

It was realized that a great deal of the manipulative work then being done was required only by the specialist: it was a tool which the ordinary boy would never use—as far as he was concerned, its mastery was a valueless possession. Having no special mathematical gifts, he found the work laborious, meaningless, and uninspiring.

For those who were to study trigonometry and the calculus such work might be necessary, but he had no hope of reaching these dizzy heights. He found the foothills hard enough climbing; when he had mastered short multiplication, long multiplcation awaited him; after short division came long division; after simple fractions there were complicated fractions, terrific in their monstrosity. All these must be conquered before he reached quadratics—in geometry, the *pons asinorum* met him on the threshold—before Book III with its attractive circle loomed Book II, while similar triangles were far away in the clouds.

In brief, the general object of the reform was to make it possible for all boys of seventeen to leave school with some knowledge of trigonometry, mechanics, and the calculus: the reform, in fact, was to make a syllabus and to adopt a line of treatment which was avowedly to be framed with a view to the needs of the ordinary boy and not solely for the specialist.

The general character of the changes that have been made are summarized in the following paragraphs:

Arithmetic. Stress has been laid upon the use of rough checks, upon the necessity of giving results to the degree of accuracy justified by the data, and upon the meaning and use of significant figures. Commercial arithmetic has been reduced and artificial problems eliminated: less time has been given to the subject and possibly accuracy has suffered.

Algebra. Greater care has been taken to explain the fundamental rules of algebra and to avoid the rule of thumb methods that tend to ruin the understanding of this subject. All the heavy manipulative work in simplifying fractions, in solving

equations by special methods, and in proving identites has been abandoned, and most boys of sxteen now have some acquaintance with the binomial theorem.

There has been a general loss of manipulative skill, but a greater appreciation of algebraic form; the work is now rarely mechanical, and shows more thought. *It is found that the manipulative power can easily be acquired at a later stage by those who need it.*

Geometry is started earlier—the theorems on parallels and congruent triangles are taken as facts, a wider range of geometry is covered, the properties of similar figures are introduced at an early stage. A further step in the reform of geometry is indicated by Professor Nunn's suggestions for basing the parallel theorems on the properties of similar triangles. Less time is given to the learning and reproduction of propositions.

Geometry has become vastly more interesting, and boys have gained considerably in the power of applying geometrical knowledge to the solution of riders.

Numerical Trigonometry is now learnt by nearly all, and a great proportion of boys do a fairly complete course of plane trigonometry omitting most of the identities which used to loom so large in the earlier treatment.

The Calculus is being read by a steadily increasing number of boys—at Harrow there are 100 boys learning calculus and at most public schools one would find at least as many.

In Mechanics our progress towards the ideal of Professor Perry is slow. An increasing number of boys learn mechanics, which is begun at an earlier stage, as part of the mathematical curriculum, but there are still many schools where boys leave without having acquired any useful knowledge of the principles of mechanics.

The more modern text-books treat the subject from an experimental point of view and use graphical methods freely to link the treatment of dynamics with the methods of the calculus, but there are still many teachers who neglect all experimntal work either from lack of time or owing to the difficulty of finding a place to do it in.

The poundal, in spite of Professor Perry's fulminations, still exists, and although daily losing strength shows surprising vitality in some parts of the country judging by the number of

candidates who in a recent examination gave the K.E. of a car of $1\frac{1}{2}$ tons moving at 30 miles per hour to be some millions of foot-poundals correct to the nearest foot-poundal with sometimes a few decimal places thrown in as well.

The claim that mathematics should appear to have a relation to life and not be a school mind-training exercise has been to a large extent secured, and it is not to be questioned that boys are far more interested in the subject than they were before. This aspect of the reform underlies the introduction into some schools of the mathematical laboratory, which formed the subject of an article in the February number of *The Journal of Education.*

The value of this work can hardly be over-estimated. It is clear that there is a very essential difference between finding the area of a surface whose dimensions are given in a text-book and one whose dimensions have to be measured. In the latter case there is a clear idea of the limitation in the degree of accuracy to which the data have been found, and it is easy to realize to what degree of accuracy the answer can be relied upon as correct: it is possible, for instance, to convince boys, when their measurements are correct to three significant figures only, that the last four of the seven they are accustomed to produce as the result of their calculations, are worthless.

There is again the aditional advantage that a mere inspection of the object in their hands will give a rough check of the accuracy of their result and there is often a second way of carrying out an experiment which will provide a check upon the first.

Added to this, there is the training afforded by the writing out, in concise and intelligible English, of an adequate account of what has been done, thus bringing welcome help to the science staff who complain so frequently of the deficiencies of schoolboys in this respect.

When the object of a reform is to benefit the average boy it is obviously necessary to secure that in so doing no harm is done to the more able student; but there is little need for anxiety on this account—the boy with mathematical gifts will flourish whatever scheme is adopted. At a large school he is soon marked out for special treatment, and although at first he is unodoubtedly inferior to his predecessors in manipulative skill he rapidly acquires this as soon as he begins to specialize.

The success of the reformed methods must depend very largely

upon the quality of the teachers. The old ways were largely stereotyped; they did not encourage individuality: the old "drive course" was evolved by a long tradition as the quickest way of learning a definite amount of mathematics, and a long tradition is seldom wrong in achieving a definite aim.

The new system requires for its success good teaching; the old would have kept its head above water however indifferent the teacher. Given a good teacher the results under the new scheme are immensely superior to the old, but with a poor teacher, old fashioned or badly trained, the new ideas cause worse failures than before. If low divisions or sets are to be taken by non-mathematicians the new methods are doomed to failure; for they require a large reservoir of ideas and knowledge.

It must also be remembered that although the reforms have been adopted with enthusiasm by a certain section of mathematical masters, schools are still largely staffed by men of the older generation, some of whom are not to be weaned from the habits of a lifetime, by any voice, however seductive. The conservative nature of our teaching is surprisingly shown by the small effect the reforms have had upon the sales of the older, well-established text-books. Although new books have appeared in shoals and their circulation has been considerable, the sale of the older books in many cases has scarcely been affected.

From the universities there still comes the complaint that boys arrive without any knowledge of mechanics, and there is no doubt much room for improvement in this respect, but it is still more surprising to find that even amongst the modern universities there are cases in which no encouragement is given to the work of reform.

The universities can cooperate with the schools in the work of reform by setting examination papers on modern lines and by supplying schools with well-trained and enthusiastic teachers. Valuable work is being done by the department for the training of teachers at Oxford and by the Board of Education in providing holiday courses for teachers who wish to keep in touch with modern developments, but the number of teachers reached in this way is comparatively small.

The non-specialist is now introduced to a great variety of mathematical topics which do not require much technique but whose value lies in their ideas. These do not lend themselves to

examination; there is sometimes very little to show which can be put to that test, but the interest of the boys has been aroused and kept alive at a time when, under the old *régime*, it was being smothered. For instance, in one of our public schools boys in the higher classical forms are allowed to choose a branch of mathematics which has to some extent appeal to them, some read Durell's "Modern Geometry," one reads the section on navigation and another the section on statistics in Nunn's "Algebra," others choose mechanics, probability, nomography, and they work quite independently with very satisfactory results.

I began this article by showing that the reformers who attacked Euclid were not breaking with a tradition which had existed for centuries, and I will conclude with two experiences which tend to show that our more recent reformers may not be such revolutionaries as our conservative friends would have us believe.

At a meeting of mathematicians a few years ago a paper was read indicating the modern methods of teaching the calculus. Upon its conclusion a venerable gentleman arose, and after expressing his approval of the methods indicated, stated that he was surprised to hear that they were considered to be modern, for to his knowledge they were identical with those used in teaching the calculus to the father of the gentleman who had read the paper!

And here is a remark made by a lady whose life has been spent in touch with public schools, and for whose opinion and judgment I have the greatest respect. She referred to the work of some small boys who had come to her for help in their mathematics and concluded by saying: "When I was a girl I was taught to understand the rules which were given and to work intelligently and with common sense: nowadays boys seem to have no idea of the reasons for what they are doing, but simply work by rule of thumb."

As on many previous occasions we learn that there were great men before Agamemnon, and there have always been reformers amongst us, but it is only in recent years that their views have met with recognition, and there is evidently still much work to be done before they meet with general acceptance.

THE NOTION OF LIMIT

By PROFESSOR DUNHAM JACKSON
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Among all the topics contained in a school course in geometry, those from which the average pupil derives the least satisfaction are probably the ones involving the notion of limit. Nowhere else do familiar words range themselves in sentences in which it is so difficult to grasp any definite meaning. The mention of the word "limit" calls forth in his consciousness little more than a vague feeling of bewilderment. When the pupil finds that the theory of infinite series is based directly on the notion of limit, he concludes at once that infinite series are beyond his comprehension. And if he is told later that the calculus deals with limits first, last, and all the time, he is likely to doubt in his heart whether the calculus is really understood by anybody at all.

The fact is, that the notion of limit requires a considerable amount of analysis and explanation, either from the lips of the teacher or in the mind of the pupil, to make it perfectly clear. It is as complicated to learn as riding a bicycle or swimming—and as simple when once learned. The course in geometry is too full of other things to leave room for an adequate treatment of limits; and the inadequacy of the treatment makes a more lasting impression than its content. There is something to be said, then, in favor of making it a special order of business to spread out on paper an account so full as to appear redundant to a student who has once grasped the idea. The beginning of such an account is attempted in the following pages.

The occurrence of limits in geometry is not the simplest exemplification of the idea; it is accompanied there by a large amount of accessory material, involving some additional difficulties. It is easiest, perhaps, to begin with an illustration from infinite series. We say that the series

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ \cdot\ \cdot\ \cdot$$

has 2 for its sum. This means, according to common understanding, that the sum of a large number of terms of the series is nearly equal to 2, the approximation becoming better and better

as more and more terms are taken. If the notation s_n is introduced for the sum of the first n terms, so that

$$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}},$$

the same meaning is conveyed by saying that s_n approaches 2 as a limit when n increases without restriction. In the strictest technical language, in fact, *the statement that the series has* 2 *for its sum, is equivalent to the statement that s_n approaches* 2 *as a limit when n becomes infinite.* The point is in appreciating thoroughly what is meant by the use of the term "limit."

In decimal notation, the successive numbers s_n are the numbers 1, 1.5, 1.75, 1.875, 1.9375, . . . It is clear that the first figure after the decimal point will always be a 9, from s_5 on. This can be expressed by saying that all the numbers s_n, from s_5 on, differ from 2 by less than 0.1, or, to anticipate the notation that will be used generally, *if ε_1 denotes the number* 0.1, *there is a number $N_1 (= 5)$ such that s_n differs from* 2 *by less than ε_1 for all values of $n \geqq N_1$.* Let ε_2 stand for another small number, say $\varepsilon_2 = 0.01$. If N_2 is taken equal to 8, it can be stated that s_n differs from 2 by less than ε_2 for all values of $n \geqq N_2$. Similarly, if $\varepsilon_3 = 0.001$, say, and $N_3 = 11$, s_n differs from 2 by less than ε_3 whenever $n \geqq N_3$. The truth of each of these statements can be immediately verified by computation. It is then just one more stage in generalization, calling for no great effort of the imagination, to say:

If ANY *positive number ε is assigned, there will exist a corresponding number N such that s_n differs from* 2 *by less than ε whenever $n \geqq N$.*

The assertion that s_n approaches 2 as a limit implies all this. So much will scarcely be doubted by anybody who has the patience to hold his attention upon it. The difficulty, for many students, comes with the addition of a supplementary clause, no less indispensable to a clear understanding of the matter. The statement about the approach to a limit means what has just been explained, *and it does not mean anything more.* It is concerned, and is concerned exclusively, with the common-sense numbers s_n, calculated by the processes of primary-school arithmetic. There is no mystical addition of *all* the terms of the series at once, no supernatural "infinitieth" term in the progression.

The general definition corresponding to the plain language that has been used above in describing a specific example is complete and sufficient. It may take an effort of faith to believe this, but such faith will have its reward. There are people who seem driven by a kind of emotional need to reach out after an undefinable and extraneous infinity, here and elsewhere in mathematics. The pursuit of this craving, whatever else it may bring, is not the way to clear insight and plain understanding.

Take for another example, very slightly varied from the first, the series

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots$$

Let s_n now stand for the sum of the first n terms of this series,

$$s_n=1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots+\frac{(-1)^{n-1}}{2^{n-1}}.$$

The assertion that the series has $\frac{2}{3}$ for its sum means that, if any positive number ε is assigned, there will exist a corresponding number N, such that s_n differs from $\frac{2}{3}$ by less than ε whenever $n \geqq N$. That is, when a value of ε has been specified, and the corresponding number N has been ascertained, it can be promised with certainty that *if you compute* (correctly) any one of the numbers s_n that you please, provided only that the subscript n is large enough, *you will get* a number differing by less than ε from the value $\frac{2}{3}$. In one respect the situation here differs from that in the first example; the difference $\frac{2}{3}-s_n$ may be either positive or negative, according to the value of n. For this reason it is convenient, in formulating a general definition, to use the symbol for the absolute value of a quantity, its numerical value regardless of algebraic sign:

The statement that a variable s_n approaches a limit A as n
The statement that a variable s_n approaches a limit A as n
becomes infinite (in symbols, $\lim_{n=\infty} s_n = A$) means that if any number ε is assigned, there will always exist a corresponding number N, such that $|A - s_n| < \varepsilon$ whenever $n \geqq N$.

This may seem like a rather long definition, at first glance. As time goes on, the fact that it is long will appear of slight consequence in comparison with the fact that it is long enough. It has to be modified and generalized to meet other situations, to be sure, but within the range of its application it is final and

complete; it really tells the story. The requisite theorems about limits can be proved by means of it, if you will take the necessary pains, as plainly and conclusively as any theorems in geometry or algebra. It is not pretended that any form of words can convey a full appreciation of the meaning of limits all at once. Practice is needed for that, and experience, more practice and experience than can be gained in a year of high school or university. But the definition is a beginning. Anybody who does the best that he can with it when he first encounters limits, and comes back to it from time to time later, whether in the way of formal study or not, will find that he has a foundation on which he can build securely.

For a geometric illustration, consider the statement that the perimeter of a regular polygon inscribed in a circle approaches the circumference of the circle as a limit when the number of sides of the polygon is increased without restriction. Let the length of the circumference be denoted by C, and let p_n denote the perimeter of a regular polygon of n sides inscribed in the circle. Since p_n is always less than C, there is no need for the use of absolute value signs. The assertion is that if ε is any positive quantity, however small, there will always exist a corresponding number N, such that $C - p_n$ is less than ε for all values of $n \gtreqless N$. There is no need of bringing any element of *time* into the discussion, of thinking of the various polygons as constructed in order, and successively passed over and forgotten. The quantities p_n are numbers which can be calculated and placed on record in any order and at any time that may be convenient. The essential thing is that when you calculate them, or whenever you come back to look at them, you can be sure that all those belonging to polygons with N sides or more will differ from C by less than ε. Of course, it may not be easy actually to calculate the numerical value of the N corresponding to a given ε, in this problem or in others of the same sort; the proof will ordinarily consist in showing by some more or less indirect means that such an N must necessarily exist. And it has been mentioned already that when limits occur in geometry they are accompanied by other difficulties, such as questions of the definitions of length and area and the axioms pertaining to them, or the meaning of irrational quantities—difficulties which do not properly belong to the notion of limit itself.

In the study of the latter topic, much puzzling has been done at one time or another over the statement of the principle that "if two variables are constantly equal, and each approaches a limit, the limits are equal." The critical pupil says: "If the two variables are constantly equal, why aren't they the same variable? What is the use of trying to prove that something is the same as itself?" The first question is a fair one; the second covers a confusion of ideas which has to be cleared up by an answer to the first. It is only a matter of words whether you speak of two variables that are constantly equal, or of a single variable; at any rate, it is a matter that does not touch the main point at issue. The same essential idea is expressed by saying that "a variable cannot approach two different limits at the same time," or, to avoid the conversational and possibly confusing use of the word *time*, "A variable depending upon a subscript n (in such a way as to be completely determined when the value of n is specified) cannot approach two different limits as n becomes infinite." The longer statement corresponds to the definition of limit that has been formulated; like the latter, it has slightly different forms for use in different situations.

If it be contended still that the alleged "principle" is trivial, it is possible now to give an intelligible answer. If you have thought about limits in a general way, without any formal definition, until you feel that you understand them well enough for your purposes, the truth of the "principle" is no doubt obvious, and to the extent that the treatment is left informal it is a matter of choice how much shall be put in the form of proof and how much shall be regarded as axiomatically evident. But if you are interested in seeing that the treatment of limits can be made as definite as that of other topics in mathematics, it is a part of the game to show that the property in question follows by logical necessity *from the definition*, is contained in the definition by implication, and does not have to be drawn from a supplementary store of miscellaneous impressions.

Finally, it will be noticed that there has been no statement anywhere in this paper to the effect that a variable cannot become equal to its limit. It is true that the most familiar variables that approach limits, whether in geometry or elsewhere in mathematics, do not attain their limits, but that is an incidental and not an essential circumstance. If the difference be-

tween variable and limit happens to be equal to zero, it is certainly less than ε, and that is all that is required. Suppose, for example, that a pendulum is set swinging, and that the vibrations are allowed to die out gradually in consequence of the resistance of the air. Under a suitable physical assumption as to the way the resistance works, it is possible to deduce a mathematical expression describing the motion completely. That is, if y is the angle which the string makes with the vertical, reckoned as positive on one side and negative on the other, and if t is the elapsed time from the beginning of the motion, it is possible to obtain a formula giving y in terms of t, so that the exact position of the pendulum at any instant can be calculated. Now the fact that the vibrations gradually become imperceptible means that the value of y comes to remain very near to zero. It is certainly reasonable to say under these circumstances that y approaches zero as a limit. In fact, such a statement is technically accurate. It is true that the definition of limit given above has to be modified in detail for the case of a continuously changing variable, as distinguished from one depending on a subscript which takes on only integral values, but the central idea remains the same. Under the definition, and for all purposes of exact discussion, the angle y in the pendulum problem does approach zero as a limit when t becomes infinite, regardless of the fact that it become exactly equal to zero again and again, as the pendulum swings through the vertical.

The treatment given in these pages is of course a mere beginning, perhaps rather a preface than a real beginning at all. It would be ample material for another paper just to show how the central idea works out in the applications that occur in elementary geometry.

MATHEMATICS OF THE CALCULATING MACHINE

By L. LELAND LOCKE

II. Automatic Operations[1]

In most power driven machines the set-up device, usually a set of slides or a keyboard, is designed to set the selector mechanism is such an operative position that one revolution of the main shaft, by hand, spring, or electric power, will serve to register the number set up or combine it with a number already in the machine. There are exceptions, as in the case of machines which involve a mechanical Pythagorean table.

In a multiplication process, as 546 multiplied by 23, the number 546 is set up and a single turn of the main shaft records 546 in the registering dials. Two more turns in succession serve to complete the multiplication by 3. The 546 may then be set one place to the left and turned in twice, thus completing the multiplication by 23. To save this resetting of the 546, the registering dials are mounted in a carriage which, as a whole, is shifted one place to the right, thus bringing the former tens dial over the 6 on the keyboard. In theory it makes no difference which is moved with respect to the other, the register dials or the selector mechanism.

Multiplier-quotient Dials. To record the 3 revolutions followed by the 2 revolutions with the carriage in the shifted position, a set of auxiliary dials is mounted in the carriage. (This set is sometimes mounted in the frame of the machine, with the counting finger moving with the carriage.) When the carriage is in the initial position with the units registering dial directly over the unts column of the keyboard, the right hand or units dial of the auxiliary set is over a counting finger, or plunger, or a one-thread worm which trips the dial from 0 to 1 with the first complete rotation of the main shaft. This turn with the following two turns record 3 on this dial. When the carriage is shifted to the right the second or tens dial of the auxiliary set is located over the counting finger and 2 turns are recorded in this position. Thus 3 turns in ones place and 2 turns in tens place are

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[1] A preceding article on this subject appeared in Vol. XV, No. 7, November, 1922.

recorded, that is, 23 is recorded in the multiplier dials. Similarly, the number of successive subtractions in a division process is recorded in the same dials which here serve as quotient dials. The multiplier-quotient dials may or may not be equipped with a carry mechanism. The above process is shown schematically in five steps, in the first three of which the carriage is in the original position and in the latter two steps the carriage has been shifted one place to the right. See Fig. 1.

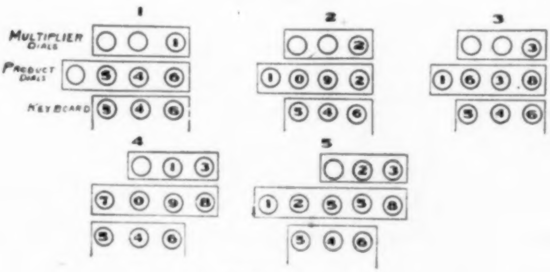

FIG. 1

The multiplier-quotient dials indicate the number of rotations of the main shaft for each position of the carriage.

Automatic Operations on the Calculating Machine. In key or power driven machines the operations of addition and subtraction are single step processes, and from that view may be said to be automatic, the operator merely setting up and registering the number in the set-up device. Multiplication and division not only require the repetition of the addition or subtraction but also require the shifting of the operand from lower to higher orders or vice versa. To be completely automatic a machine must be equipped with devices which shall stop the machine when the proper number of rotations has been executed, shift the carriage, and continue this succession until the process is complete.

For multiplication such a machine is comparatively simple in principle, since the number of rotations for each order is predetermined. A set-up device is provided for the multiplier, which has the double function of stopping the machine when the proper number of rotations has been made for one order, shifting the carriage one place to the right or left, and repeat-

ing these functions for each order of the multiplier. Such a set-up device may consist of: 1° a slide or set of keys numbered from 1 to 9. This single set would require a resetting for each order of the multiplier. 2° a full bank of keys sufficient to set the whole multiplier. 3° a dial set-up which takes care of the whole multiplier. Let it be required to multiply 5896 by 437. The multiplicand 5896 is set up on the keyboard, and the multiplier 347 in the multiplier set-up.

A typical construction might be described as follows: A series of stops is provided such that when the carriage has its units dial over the 6 in units place on the keyboard, an operative piece on the carriage is over the first stop which is set at 7. This stop allows the main shaft to revolve 7 times, then releases the carriage which shifts one place to the right bringing the operative piece on the carriage over the second stop which is set at 4, and the process continues. After the multiplier and multiplicand have been set up the machine requires no further attention from the operator until the multiplication is completed. In practice it is immaterial whether the machine is built to begin with the right or left hand figure of the multiplier. For the sake of speed it may be desirable to have the machine so constructed that the operation begins as soon as one figure of the multiplier has been set up, the multiplicand being set first.

Recent designs would seem to indicate that automatic multiplication is not regarded as of sufficient importance to warrant the added complexity of mechanism. Automatic multiplication requires two complete set-up devices. In addition there is a decided loss in possible short cuts. In multiplying by 398, the automatic performs 20 rotations, while in the non-automatic the same multiplication may be performed with 6 turns, multiplying by 400 with 4 positive rotations and subtracting twice with two negative rotations. In multiplication the number of rotations in each order is predetermined. This number may be noted, either by counting or by observing the multiplier dials, with a machine operating as rapidly as the newer electric models (five rotations per second). The non-automatic requires concentrated attention during the process.

Automatic Division. Division presents a different situation. The operation begins on the left and is performed by successive

subtractions, the number of subtractions in each order being recorded in what may now be called quotient dials. The important distinction in contrasting division with multiplication is that in the former the number of rotations is not predetermined. After each individual subtraction, the question of whether or not another subtraction is possible must be determined by a comparison of the divisor with the partial dividend remaining at this point. Let the required division be: 27)78. The first subtraction accomplished by one turn of the crank may be represented thus, $27)\overset{1}{51}$. The second turn produces $27)\overset{2}{24}$. The 24 being less than the 27, another subtraction is not possible. Such a comparison is not a mere matter of counting as in multiplication but requires an individual judgment for each revolution. In the construction of a machine for automatic division the problem presents itself in the following fashion. At no stage of the process is there mechanical means for this comparison and therefore no device to stop the machine and shift the carriage when the requisite number of turns has been reached. The design of a machine for automatic division is interesting in the mathematical relations involved as well as in the ingenious methods of realizing these processes mechanically. As a preparation for the problems involved in division consider the action of the dials and carry mechanism in a two-way adding and subtracting machine, that is, a machine in which the dials rotate in a positive direction for addition and in a negative for subtraction. As the dial in units place is turned in a positive direction from the position 9 to the next position 0, the dial in the next higher order is turned from the digit it holds to the next higher digit. If the dials show 0009, adding 1 in the units place will change the reading to 0010. If each dial is set to register 9, and 1 is added in units place, the units dial will turn from 9 to 0, and 1 will be added to the tens place which changes it from 9 to 0, etc. This carry will run progressively throughout all the dials which have registered 9. As the extreme left hand dial changes from 9 to 0, the "kick" which produced the carry with other dials is not needed as there is no dial on its left. It may then be utilized to ring a bell, or otherwise, as indicated later on. (Mechanically such a trip or kick may be taken off for any purpose

at any point in the series without detriment to its other use as a carry.)

Consider motion in the opposite direction. If the dials are reversible and the machine is equipped with a reversible carry, as the units dial is turned in a negative direction as in subtraction and passes from 0 to 9, the next or tens dial is turned backward from the digit it registers to the next preceding digit. If each dial registers 0 and 1 is subtracted in units place, each of the dials in ascending order will turn backward from 0 to 9, the kick from the last carry being utilized as before to ring a bell, or otherwise as will be shown. If in the example above, $\begin{array}{r}2\\ 27)24\end{array}$, 27 is again subtracted, the machine will register, $\begin{array}{r}3\\ 27)99999997\end{array}$. The carry from the lefthand 9 rings a bell and warns the operator that one too many subtractions has taken place. This is corrected by reversing the direction of rotation, adding 27 by a positive turn of the dials. The carriage is then shifted to the left and the division is continued. No observation of the quotient dials nor comparison of divisor with partial dividend is necessary from the beginning to the end of the process, the operator merely attending to the signals for reversing and shifting. Upon the principles above are based the designs of the automatic machine. If instead of using the last carry in subtraction, when the 9s appear to ring a bell, it is caused to trip a mechanism which reverses the machine, and the carry which results from clearing the machine of 9s with the first positive rotation is caused to shift the carriage and reverse the machine, the process becomes entirely automatic. The problems of realizing the stopping of the machine smoothly and the gradual taking up of the load after reversal are for the designer and in no way concern the mathematical principles involved.

In a machine in which the dials rotate in but one direction and with which subtraction is performed by overaddition, the problem of automatic division presents a different aspect. A brief explanation and a diagram of the proportional racks of the Mercedes-Euklid will be given as presenting an ingenious method by which an easy shift from addition to overaddition is accomplished without the codigit resetting of the keydrive ma-

chines. For the sake of clearness some liberties have been taken with the disposal of the parts. See Fig. 2.

A set of racks slide horizontally, the motion being transmitted by a crank operated by shaft C. At the opposite end of the racks is a slide within which a pin may be moved by a lever from

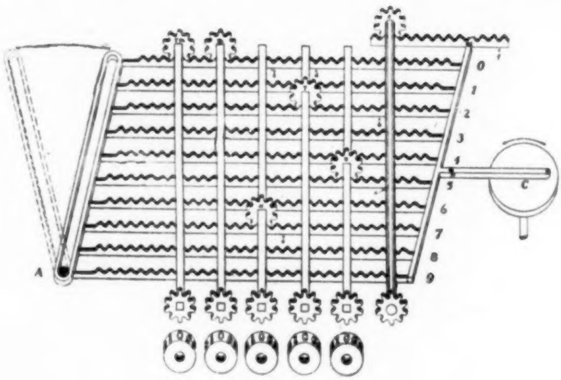

FIG. 2

position A to position B. When the pin is in position B the rack numbered 0 is locked in position and each of the other racks move through a distance proportional to its number with a rotation of the crank. Mounted above these racks is a set of pinions sliding on square shafts which transmit the motion of the pinions to a set of other pinions in front of rack 9, and through them to the dials. If the pin occupies the A end of the slide, as here pictured, rack 9 is locked in position and each of the other racks move a distance proportional to its distance from rack 9. Rack 8 moves one tooth, 7 moves two teeth, 6 three teeth, or in general a rack moves the codigit of its rack number.

Above the square shafts, slots are cut in the top plate of the machine, which carry numbers corresponding to the rack numbers. Through these slots, which correspond to the orders, beginning at the right, ones, tens, hundreds, etc., the pinions are set over the racks by means of forks passing down through the slots and attached to the pinions. With the pinions set as in the cut, the units pinion is set at 5, the tens pinion at 2, and the hundreds pinion at 7, while the thousands and ten-thousands pinions are each set at 0. If the pin is at B, a turn of the crank will register 725. If the pin is set at A, as indicated in the cut,

rack 9 is locked and the register for one turn of the crank is 99274. (Small arrows in the cut indicate the extent of the rack travel.) A pinion X at the right of the units shaft, and which is not connected with a dial, makes a complete revolution, with the pin in position A. (In the actual design of the machine the small rack which operates pinion X is attached to the O rack, but having ten of its teeth equal to 9 teeth of rack 0.) This complete rotation of pinion X produces a carry-1 to the units dial, thus completing the recording of the supplement of 725, or 99275. Throwing the lever which moves the pin from B to A changes the process from addition to that of addition of the supplement which is equivalent to the subtraction of the number set

During the return travel of the racks the pinions in front of rack 9 are out of mesh with the dials and during this period when the dials are free, the carry mechanism operates, giving rise to what is called delayed carry.

The quotient dials to take of this shift from addition to over-addition may be designed as follows: As the main shaft rotates in but one direction, the quotient dials will necessarily rotate in either direction as demanded by the situation and will be equipped with a reversible carry. A lever shift determines the direction of rotation of these dials. In multiplication they rotate in a positive direction. In division they will rotate in a positive direction during the subtraction process or with the pin in position A. The shift of the pin from A to B will reverse the direction of the dials so that they turn in a negative direction during the addition necessary in the division process.

The formula involved in automatic division on a machine of this type is derived as follows:

D is the dividend, *d* the divisor, *a* the first quotient figure, *b* the second quotient figure, and *r* the partial dividend after two divisions have been made.

$$\frac{D}{d} = a \cdot 10^n + b \cdot 10^{n-1} + \frac{r}{d} = (a+1)10^n - (10-b)10^{n-1} + \frac{r}{d}$$

In the division first described, which will be designated as *A*, subtractions take place until the correct number *a* is overrun by 1, at which point the machine is reversed and this 1 is destroyed by one addition, after which the carriage is shifted to the next order. With the latter type of machine, which will be indicated

by B, the cycle of division covers two orders, in the first of which subtraction takes place and in the second the operation is addition. The divisor is subtracted $a + 1$ times in the first order and added $10 - b$ times in the second, as indicated in the second form of the above formula. When the $a + 1$ subtractions have been made, the succession of 9s, the result of an over run in supplemental subtraction, shifts the carriage and reverses the machine to addition. The quotient dial reads $a + 1$, or 1 more than the correct figure. With the reversal, the direction of rotation of the quotient dials is reversed in direction and the first addition turns the second quotient dial back from 0 to 9. The dials being equipped with a two-way carry, the backward carry is transferred to the first dial, changing the reading from $a + 1$ to a, the correct reading. In the second position of the carriage the number of additions necessary is $10 - b$, but since the quotient dials are now rotating backward, the correct reading b appears on the dial. When this reading has been reached, the restoration of the 9s to 0s sets in motion another reversal and carriage shift and a new cycle begins.

The machine readings in a division example at each stage of the process are given, first in a machine of type A, and second for type B.

Type A

Example: 725)17767(24
367

1st Step: Subtract 725 3 times.

Keyboard Divisor	Accumulation Dials Dividend	Quotient Dials Quotient
725	17767	0
	10517	1
	3267	2
	**999996017	3

2nd Step: Reverse and add 725 1 time.

725	**000003267	2

3rd Step: Shift carriage and subtract 725 4 times.

725	000003267	2
	2542	21
	1817	22
	1092	23
	367 remainder	24 quotient

Type B

Example: 725)17767(24
367

1st Step: Add 99999275 3 times.

Keyboard	Accumulation Dials	Quotient Dials
9999999275	17767	0
	10517	1
	3267	2
	***99999996017	3

2nd Step: Reverse, shift carriage and add 725 10-4 times.

725	99999996017	3
	99999996742	*29
	99999997467	28
	99999998192	27
	99999998917	26
	99999999642	25
	**00000000367 remainder	24 quotient

* Reverse carry to tens place.

** First end carry off which rings the signal bell or sets in motion those mechanical movements which make the machine automatic.

*** First non-carry off which has the same property as **.

THE UNIFICATION OF MATHEMATICAL NOTATIONS IN THE LIGHT OF HISTORY[1]

By PROFESSOR FLORIAN CAJORI
University of California

Uniformity of mathematical notations has been a dream of many mathematicians—hitherto an iridescent dream. That an Italian scientist might open an English book on elasticity and find all formulæ expressed in symbols familiar to him, that a Russian actuary might recognize in an English text signs known to him through the study of other works, that a German physicist might open an American book on vector analysis and be spared the necessity of mastering a new language, that a Spanish specialist might relish an English authority on symbolic logic without experiencing the need of preliminary memorizing a new sign-vocabulary, that an American travelling in Asiatic Turkey might be able to decipher, without the aid of an interpreter, a bill made out in the numerals current in that country is indeed a consummation devoutly to be wished.

Is the attainment of such a goal a reasonable hope or is it an Utopian idea over which no mathematician should lose precious time? In dealing with perplexing problems relating to human affairs we are prone to look back into history, to ascertain if possible, what light the past may shed upon the future and what roads hitherto have been found impassable. It is by examining the past with an eye constantly on the future, that the historical student hopes to make his real contribution to the progress of intelligence.

The admonition of history is clearly that the chance, haphazard procedure of the past will not lead to uniformity. The history of mathematical symbolism is characterized by a certain painting representing the landing of Columbus; the artist painted three flags tossed by the breeze, one east, one west and one south—indicating a very variable condition of the wind on that memorable day. Mathematical sign-language of the present time is the result of many counter currents. One might think that a perfect agreement upon a common device for marking decimal fractions could have been reached in course of the cen-

[1] Vice-Presidential address before Section L of the A. A. A. S., December 29, 1923.

turies. Yet such is not the case. During the sixteenth and seventeenth centuries, when the algorithm of decimal fractions was being created, one might reasonably expect much experimentation on symbolism. That annoying diversity should prevail down to the present time is truly indicative that any casual procedure will not lead to the desired goal. At present, the notation 2.5 means one thing to an American and another thing to an Englishman; to the American it signifies two and five-tenths, to the Englishman it stands for two times five or ten. In other countries the comma, placed along the lower line of type or sometimes placed high as if an apostrophy, plays the role of decimal separatrix. No doubt some will be astonished to hear that in recent books, printed in different countries, one finds as many as ten different notations for decimal fractions. Oh goddess of chaos, thou art trespassing upon one of the noblest of the sciences. The force of habit conspires to the perpetuity of obsolete symbols. In Descartes' Geometry of 1637 appeared the modern exponential notation for positive integral powers. Its ideal simplicity marked it as a radical advance over the older fifteenth and sixteenth century symbolisms. Nevertheless antiquated notations maintained their place in many books for the following fifty or seventy-five years. Other cases similar to this could be cited. In such redundancy and obsoleteness, one sees the hand of the dead past gripping the present and guiding the future.

Students of the history of algebra know what a struggle it has been to secure even approximate uniformity of notation in this science, and the struggle is not yet ended. The description of all of the symbols which have been suggested in print for the designation of the powers of unknown quantities and also of known quantities would fill a small book. Rival notations for radical expressions unnecessarily complicate the study of algebra in our high schools today. And yet ordinary algebra constitutes the gateway to elementary mathematics. Rival notations embarrass the beginner at the very entrance into the field of this science.

This confusion is not due to the absence of individual efforts to introduce order. Many an enthusiast has proposed a system of notation for some particular branch of mathematics, with the hope, perhaps, that contemporary and succeeding workers in the

same field would hasten to adopt his symbols and hold the originator in grateful remembrance. William Oughtred in the seventeenth century used over 150 signs for elementary mathematics—algebra, geometry and trigonometry. Many of them were of his own design. But at the present time his sign of multiplication is the only one of his creation still widely used; his four dots for proportion are redundant and obsolescent; his sign for arithmetical difference $\backsim$ has been employed for wholly different purposes. In France Hérigone's writings of the seventeenth century contained a violent eruption of symbols, now known only to the antiquarian. The Hindenburg combinatorial school at the close of the eighteenth century in Germany introduced complicated symbolic devices which long since passed into inocuous desuitude. The individual designer of mathematical symbols, looking forward to enduring fame, is doomed to disappointment. The direction of movement in mathematical notation, when left to chance, cannot be predicted. When we look into the history of symbolism, we find that real merit has not always constituted the determining factor. Frequently wholly different circumstances have dominated the situation. The sign of division $\div$, now used in England and America, first appears in a Swiss algebra of 1659, written in the German language, which enjoyed no popularity among the Germans. But an English translation of that book attracted wide attention in England and there led to the general adoption of that sign of division. The outcome of these chance events was that Great Britain and America have at the present time a sign of division used nowhere else; the Leibnizian colon employed for division in continental Europe, is not used for that purpose in the English speaking countries. Another example of the popularity of a book favoring the adoption of a certain symbol, is Byerly's Calculus which secured the adoption in America of J. E. Oliver's symbol $\doteq$ for approach to a limit.[1] William Oughtred's pupils (John Wallis, Christopher Wren, Seth Ward and others) helped the spread in England of Oughtred's notation for proportion which attained popularity also in continental Europe. The Dutch admirers of Descartes adopted his sign of equality which became so popular in conti-

[1] Since writing the above I have found the use of $\doteq$ for "nearly equal to" (nahezu gleich) in the "Algebra" of A. Steinhauser, Vienna, 1875, p. 292. This date is five years earlier than the first appearance of $\doteq$ in print in the United States.

nental Europe in the latter part of the seventeenth century as almost to force out of use the Recordian sign of equality now generally adopted. The final victory of the Recordian sign over the Cartesian rival appears to be due to the fact that Newton and Leibniz, who were the two brightest stars in the mathematical firmanent at the beginning of the eighteenth century, both used in print the Recordian equality. Thus history points to the generalization that individual effort has not led to uniformity in mathematical language. No doubt many mathematicians of the present time are wondering what will be the fate of the famous symbolisms of Peano and of Whitehead and Russell. In the light of past events this system of symbols will disintegrate as did the systems of Oughtred, Hérigone and Hindenburg. Except Liebniz and Euler, no mathematician has advanced more than two ideographic symbols which have been universally adopted. The signs of elementary algebra constitute a mosaic composed of symbols taken from the notations of more than a dozen mathematicians. In the various fields of mathematics there has been, until very recently, no method of selection, save through mere chance. Consequently some symbols have several diffrent significations; also, the same idea is often designated by many different signs. There is a schism between the situation as it exists and what all workers would like it to be. Specialists in advanced fields of mathematics complain that the researches of others are difficult to master because of difference in sign language. In the eighteenth century mathematicians framed and tried to remember a simple rule for changing from the Leibnizian dx to the Newtonian $\dot{x}$. But the rule was inaccurate and endless confusion arose between differentials and derivatives.

It is clear that new forces must be brought into action in order to safeguard the future against the play of blind chance. The drift and muddle of the past is intolerable. We believe that this new agency will be *organization, cooperation.* To be sure the experience of the past in this direction is not altogether reassuring. About a quarter of a century ago a group of international workers constituted themselves a committee for the formulation of an international symbolism in vector analysis, but the movement fell through. There was a deep appreciation, on the part of the individual worker, of the transcendant superiority

of his own symbols over those of his rivals. However, this intense individualism was not the sole cause of failure. Eventually, an appreciation of the ludicrous might have saved the situation, had it not been for the disorganization, even among scientific men, resulting from the great war. The causes just named brought to naught the work of another international group of specialists who, about 1913, started out with the laudable endeavor to devise a notation acceptable to workers of all nationalities in the theories of elasticity and potential. However, somewhat more successful was a third group of reformers interested in the field of actuarial science. In this line the British Actuarial Society had been active for many years, devising a full system of notation, so clear and efficient as to meet the approval of all. With some slight modifications, the English proposals were adopted by the international congress of actuaries in 1895 and 1898. As a result of this action, many of the symbols in a very long list have been widely adopted, but as far as we can learn, they are not used to the exclusion of all other symbols. This movement marks a partial success. Perhaps some improvement in the mode of procedure might have led to a more universal acceptance of the symbols agreed upon by the international experts. Past failures have not deterred our National Committee on Mathematical Requirements from recommending that their report on symbols and terms in elementary mathematics be considered by the International Congress of Mathematicians. The preparation of standardized symbols and abbreviations in engineering is now entered upon, under the sponsorship of the A. A. A. S. and other societies.

The adoption at a particular congress of a complete symbolism intended to answer all the present needs of actuarial science finds its counterpart in the procedure adopted in world movements along somewhat different lines. In 1879 a German philologian invented a universal language, called volapuk. Eight years later a Russian invented esperanto, containing 2642 root-words. Since then about eight modified systems of universal language have been set up. It will be recognized at once that securing the adoption of a universal spoken and written language by all intellectual workers is an infinitely greater task than is that of a universal sign-symbolism for a very limited group of men called mathematicians. On the other hand the

task confronting the mathematician is more difficult than was that of the electrician who secured the adoption of an international system of electro-magnetic units. The mathematician cannot depend upon immense commercial enterprises involving large capital to exert a compelling influence such as brought about the creation and adoption of a world system of electric and magnetic units. Even the meteorologist has the advantage of the mathematician in securing international signs; for symbols describing the weather assist in the quick spreading of predictions which may safeguard shipping, crops and cattle.

The movement for a general world language is laboring under two impedimenta which the mathematicians do well to avoid. In the first place there has been no concerted, united action among the advocates of a world language. Each works alone and advances a system which in his judgment transcends all others. He has been slow to learn the truth taught to the savage by his totem, as related by Kipling, that

> "There are nine and sixty ways
> Of constructing tribal lays
> And every single one of them is right."

Unhappily each promoter of a world language has been opposing all other promoters. The second mistake is that each system proposed represents a finished product, a full-armed Minerva who sprang from the head of some Jupiter in the field of philology.

In the endeavor to secure the universal adoption of mathematical symbolism international cooperation is a *sine qua non*. Agreements by representatives must take the place of individual autocracy. Perhaps scientists are in a better position to cultivate internationalism than other groups. A recognition of that fact is embodied most strikingly in the policy of the journal called *Scientia*, edited under the leadership of the Italian Eugenio Rignano. The aim of *Scientia* is not only the synthesis in science and a viewing of the general status and interrelation of broad fields of science, but the bringing about of friendly cooperation between scientific workers in different parts of the world. *Scientia* aims to be a link, a common place of meeting, a territory belonging to no one nationality exclusively. It rests on the belief that the most rapid advance of science can be attained only by internationalizing science.

The second warning alluded to above is that no attempt should be made to set up at any one time a full system of notation for any one department of higher mathematics. This warning has remained unheeded by some contemporary mathematicians. In a growing field of mathematics the folly of such endeavor seems evident. Who is so far-sighted as to be able to foresee the needs of a developing science? Leibniz supplied symbols for differentiation and integration. At a later period in the development of the calculus the need of a symbolism for partial differentiation arose. Later still a notation for marking the upper and lower limits in definite integrals became desirable, also a mode of designating the passage of a variable to its limit. In each instance numerous competing notations arose and chaos reigned for a quarter or half a century. There existed no international group of representatives, no world court, to select the best symbols or perhaps to suggest improvements of its own. The experience of history suggests that at any one congress of representative men, only such fundamental symbols should be adopted as seem imperative for rapid progress, while the adoption of symbols concerning which there exists doubt should be discussed again at future congresses. Another advantage of this procedure is that it takes cognizance of a disinclination of mathematicians as a class to master the meaning of a symbol and use it, unless its introduction has become a practical necessity. Mathematicians as a class have been accused of being lethargic, easily pledged to routine, suspicious of innovation.

Nevertheless, mathematics has been the means of great scientific achievement in the study of the physical universe. Mathematical symbols with all their imperfections have served as pathfinders of the intellect. But the problems still awaiting solution are doubtless gigantic when compared with those already solved. In studying the microcosm of the atom and the macrocosm of the milky way, only mathematics at its best can be expected to overcome the obstacles impeding rapid progress. The dulling effect of heterogeneous symbolisms should be avoided. Thus only can mathematics become the Goliath sword enabling each trained scientist to say: "There is none like that; give it me."

TWENTY YEARS OF THE ASSOCIATION OF TEACHERS OF MATHEMATICS IN NEW ENGLAND[1]

Preliminary Steps. The need of some organization which should bring together college and school teachers of mathematics seems to have been recognized by various persons independently interested about 1900. Professor W. F. Osgood of Harvard discussed the matter with Mr. G. W. Evans of the English High School, Boston; Mr. W. T. Campbell, of the Boston Latin School; Mr. E. H. Nichols, of the Browne and Nichols School, and Mr. W. A. Francis, of Exeter. Professor H. W. Tyler of the Institute of Technology, communicated with Messrs. R. F. Curtis and W. C. Hagar, of Chauncy Hall, referring to the recent work of the Committee of the American Mathematical Society on definitions of entrance requirements, and suggesting that it might be advantageous to form an association of teachers of mathematics with a view to promoting esprit de corps among the mathematical teachers in secondary schools, to enable them to maintain more effective relations with college men and perhaps to remove the occasional misapprehension that mathematics was long since a completed subject, without present vitality or growth. The matter was doubtless in the minds of others as early, or earlier, than this, and no suggestion of personal priority is intended.

An informal conference at the Twentieth Century Club of a small group of persons interested was followed by the circulation of a printed invitation, dated March 18, 1903, reading as follows:

> It is proposed to organize an Association of Teachers of Mathematics in New England. For this purpose you are invited to cooperate with us by sending the names and addresses of any teachers under your direction who are actually engaged in the teaching of algebra, geometry or trigonometry as a main part of their work. An early reply from you will be of great service to us, and will be regarded as a favor. (Signed) E. P. Seaver.

[1] Report of a committee presented to the Association of Teachers of Mathematics in New England, May 5th.

Mr. Seaver was then Superintendent of Schools in Boston. This is the earliest document in the files of the Association. In April, 1903, the following printed letter was circulated:

"Boston, April 3, 1903.

"It is proposed to form an association of New England teachers of mathematics and others interested in mathematical instruction for the purpose of holding meetings, usually in or near Boston, to discuss questions relating to the subject matter and the teaching of elementary mathematics. Among the objects which such an association would strive to attain may be named: To give to elementary mathematics a more important place in the training and thought of the pupil, and to make the instruction in mathematics more effective and more closely related to practical affairs; to increase the teacher's interest in the science of mathematics, and thus to react beneficially on his teaching.

"Aids to these ends might consist in part, beside the papers on special topics, in the presentation and discussion of reports of committees and individual members of the society on important modern text-books and other publications, both American and foreign, of interest to teachers; and on methods of teaching mathematics here and abroad.

"A meeting will be held in the building of the Boston Latin School, corner Warren Avenue and Dartmouth Street, Boston, on Saturday, April 18, 1903, at eleven o'clock, for the purpose of taking the necessary steps for the organization of such an association, and you are invited to be present. The meeting will be addressed by Professor Thomas S. Fiske, of Columbia University, President of the American Mathematical Society. At the afternoon session a paper on 'Observational Geometry' will be presented by Mr. William T. Campbell, of the Boston Latin School, who will exhibit a collection of geometric models made by pupils of the school, and discussion from the floor will be invited.

"A copy of the proposed constitution is inclosed.

W. T. Campbell,
Boston Latin School;

F. P. Dodge,
Roxbury Latin School;

G. W. Evans,
English High School, Boston;

E. H. Nichols,
The Browne and Nichols Sch.

W. F. Osgood,
Harvard University;

J. C. Packard,
Brookline High School;

E. F. Pendleton,
Wellesley College;

H. W. Tyler,
Mass. Inst. of Tech."

The sample constitution enclosed seems to call for no special comment or quotation.

The first annual meeting was held in the following November, at the Browne and Nichols School, Cambridge, with papers on "Holzmüller's Stereometry" by Professor E. V. Huntington of Harvard and "Graphical Representation in Arithmetic and Algebra," by Mr. G. W. Evans of Boston.

In February, 1904, a mid-winter meeting was held at Springfield; and in April, the second spring meeting at the Institute of Technology.

The fourth printed program is that of the second spring meeting, April 23, 1904, at the Institute of Technology, on the general subject "Geometry," with addresses by Mr. Nichols on "The Teaching of Formal Geometry," and C. D. Meserve of Newton on "The Use of a Syllabus and Original Problems in Teaching Geometry."

A letter was addressed to Mr. Nichols, May 1, 1903, in regard to the appointment of a committee to consider the subject of a syllabus. There was incidental discussion of the question of forming local sections.

The membership list published under date of January 1, 1904, showed 170 members in all the New England States.

For the year 1903-4, Mr. Nichols was president, Professor Osgood, vice president, and Mr. F. P. Dodge, Roxbury Latin School, secretary—presumably discharging also the duties of an (unmentioned) treasurer. The other members of the first council were Miss E. K. Price, Springfield; Professor W. A. Moody, Bowdoin; Mr. W. A. Francis, Exeter; Mr. G. W. Evans, Boston; Mr. J. C. Packard, Brookline, and Professor H. W. Tyler, Boston.

The first report of the association, containing the list of officers, the secretary's report, a report on "Instruction in Observational Geometry," a summary of Mr. Nichols' paper on "Teaching of Formal Geometry," and a digest of Mr. Meserve's paper on "The Syllabus Method," was published in May, 1904.

The third spring meeting—April, 1905—included a preliminary report of the committee on "List of Theorems in Elementary Geometry" and addresses on "Arithmetical Work" and "Computation," by Professor J. F. Norris, Simmons, and Professor H. M. Goodwin, M. I. T. The beginnings of committee

reports and of addresses by speakers outside of membership of the society are significant.

In 1905 Mr. Francis became president, Professor N. F. Davis, Brown University, vice president, and Mr. G. W. Evans, secretary-treasurer. With few exceptions the association has chosen a school man as president, a college man as vice president. Mr. Evans continued as secretary until 1910. Mr. W. B. Carpenter, of the Mechanic Arts High School, became treasurer in 1907; he was succeeded by Mr. F. P. Morse in 1910, and he in turn by Mr. F. W. Gentleman of the Mechanic Arts High School in 1912. Mr. H. D. Gaylord of the Browne and Nichols School succeeded Mr. Evans as secretary in 1911. The indebtedness of the association to the self-sacrificing service of these officers cannot be overestimated.

Without discussing later programs in all details, certain salient facts may be noted in a review of them. In 1905 a new point of view is represented in addresses on the "Attitude of the Teaching Public Toward Mathematics," by Mr. J. P. Clark of Lynn, and on "The Demand of Industrial Science for Mathematics," by Mr. Lewis Saunders of the General Electric Company.

In 1906 college entrance examinations came to the fore, accompanied by a discussion of "Proposed Changes in the Teaching of Geometry and of Arithmetic." This appears to have been the beginning of the two-session programs.

In December of the same year there are committee reports on geometry, on arithmetic, on pattern examination papers, and a discussion of "Recent Examination Results at Harvard," by Professor E. V. Huntington, while Professor L. L. Conant of Worcester discusses "The Beginnings of Number"—the first of a considerable list of addresses of scientific significance.

In 1907 the practical side is again emphasized by Mr. C. F. Warner of Springfield, and Professor F. B. Sanborn of Tufts, while an imported speaker, Dr. J. T. Rorer of Philadelphia appears for the first time.

In February, 1907, the association paid its first visit to New Haven, holding a joint meeting with the Connecticut Association of Teachers of Mathematics. This was the first of a long series of mid-winter regional or missionary meetings, held at Providence, Springfield, Worcester, Manchester, Concord and elsewhere.

In December, 1907, there is another report on "Pattern Examinations" and an address by Professor Osgood on the "Solution of Numerical Equations." Singularly enough it would appear that the pattern examinations prepared by the school men aroused even more vigorous criticism from school men themselves than had been bestowed on the college examination papers, the defects of which had led to the appointment of the committee.

The program of April, 1908, raises the question: "Is the Teaching of Theory of Limits Worth While?" answered by Prof. H. E. Hawkes of Yale and Mr. Francis of Exeter; and "Do Book Problems Misrepresent Algebra?" answered by Prof. W. R. Ransom of Tufts. That for December of the same year follows with a discussion of "Loci Problems," as a result of which a special pamphlet on loci was published by the association. A question box in charge of Mr. Evans, announced in this program, was doubtless intended as a continuing institution, but subsequent references to it are meagre.

March, 1909, combines discussion of proposed changes in the teaching of arithmetic, and some changes (whether "proposed" or otherwise is not stated) in teaching geometry, with numerous participants.

In December, 1909, a meeting was held in connection with that of the American Association for the Advancement of Science. Professor I. J. Schwatt of the University of Pennsylvania, announced in the program, was unable to appear; Mr. Packard and Professor H. E. Clifford of Harvard discuss the "Perry Movement" and "Problem Work in Mathematics Instruction," respectively. This afternoon meeting was so great a success that it was decided by mail ballot to concentrate efforts on a single session.

In May, 1911, the association listens to eminent and vigorous critics of "Mathematical Teaching and Teachers" in Professor G. F. Swain of Harvard and Dr. David Snedden, State Commissioner of Education. The latter was answered by Professor Coolidge in an extempore rejoinder remarkable both for substance and for eloquence. The following December there is a return to the faithful in Professor G. D. Olds, Amherst, on "The Conduct of Class Room Exercises" and Professor Coolidge on "Le Moine's Géométrographie."

The program for March, 1911, makes the first reference to subscription for *School Science and Mathematics* in connection with the annual dues.

In April, 1912, the association, appreciating the social opportunities of the inter-session luncheon, returned to the morning and afternoon session program, and there appears for the first time the title "Book Reviews"; at the same meeting, Mr. William Fuller of the Girls' Latin School discusses the Provisional Report of the National Committee of Fifteen on "Geometry Syllabus."

In December, 1912, history of mathematics is emphasized for the first time with an address by Professor David Eugene Smith, Columbia, on "Great Problems in Secondary Mathematics" and an exhibition of lantern slides by Professor Tyler.

In May, 1913, the morning session is impartially divided among geometry, algebra, and arithmetic, while in the afternoon Professor L. M. Passano, M. I. T., discusses "Efficiency vs. the Individual." Announcement is made that THE MATHEMATICS TEACHER is a perquisite of membership.

In February, 1914, the association held its first and only meeting with the Association of Teachers of Mathematics in the Midle States and Maryland, in New York City.

In March, 1914, the morning and afternoon sessions were followed by an informal dinner at the Hotel Nottingham. the program consisting of book and magazine reviews. While the dinner is the first of those mentioned in the official notices, the practice of holding informal dinners for nearby men dates almost from the beginning of the association. The intimate relations established by these round table discussions had an important influence on the character and conduct of the association. The subsequent occasions of this character, open to the general membership of the association, have proved a valuable feature of its work.

December, 1914, comes a report on the "Mathematics of Pre-High School Grades" by Mr. W. L. Vosburgh, Boston, as chairman of a committee, and a report of a Committee on the Present Status and Welfare of Mathematics in Secondary Schools, followed by an address by Mr. H. C. Morrison, State Superintendent of Public Instruction in New Hampshire.

The program for May, 1915, also includes reports of the two committees mentioned above; the second report being focussed on the special topics "Mental Training," "Preparation of Teachers of Mathematics" and a "Minimum List of Topics in Algebra" with a discussion of the question: "Is it practicable and desirable to introduce a composite course in high school mathematics, designed to serve the needs of pupils who are not going to college; this to be followed by a more detailed preparatory course for those who are to pursue mathematics further." Professor R. W. Willson, Harvard, diversifies the program by exhibiting simple astronomical apparatus for use in schools.

In December, 1915, the title of the society was changed to the Association of Teachers of Mathematics in New England, and further reports from members of the Committee on Status and Welfare deal with the preparation of teachers of mathematics in Massachusetts high schools (Dean F. C. Ferry, Williams), "Algebra as a Required Subject" (Mr. Morse) and "A List of Topics in Algebra" (Mr. Evans). Professor H. A. Merrill, Wellesley, reports on the Napier Tercentenary.

The final report of this committee, and that on Mathematics of Pre-High School Grades were presented in May, 1916, following addresses on "Graphs in Algebra" by Mr. H. C. Barber, Boston, and "Euclid and His Works" by Professor R. C. Archibald, Brown.

In April, 1917, Professor Ernest C. Moore, Harvard, discusses "Formal Discipline"; and in the following December Professor A. J. Inglis, Harvard, speaks on "Mathematics and the Problem of Transfer," criticizing Professor Moore's position.

In December, 1918, there are reports on "A Program in Mathematics for the Junior High School" and on "Regent College Entrance Examinations."

The December, 1919, program contains the first of several reports on "Mathematical Requirements," by Mr. W. E. Downey of the Boston English High School.

In May, 1920, the association proves itself up to date by listening to a paper on "Einstein's Theory" by Professor C. L. E. Moore, M. I. T. Incidentally the program shows an Einsteinian disregard for limitaticns of time and space by combining as speakers Professor L. R. Perkins, Middlebury; Professor Chapman, Yale, and Dr. Wilmer Souder, of the National Bu-

reau of Standards. In December Mr. H. P. McLaughlin, Boston, presents "Magic in the Algebra Class." Professor C. H. Currier, Brown, discusses "Mathematics of Insurance" and Professor Joseph Lipka, M. I. T., "Graphical Methods of Computation."

In May, 1921, it enters a new field in a discussion of "Proportional Representation" by Professor Huntington. At the same meeting Mr. J. A. Foberg of the National Committee reports on "Junior High School Mathematics" and President Eliot discusses the "Teaching of Secondary Mathematics."

In December, 1921, motion pictures are introduced. Mr. Eugene R. Smith of Baltimore describes "Modern Methods in Junior High Schools" and Professor R. E. Bruce, Boston University, speaks on "Graphic Solution of Systems of Equations." There is a (second) combination dinner with the New England Association of Colleges and Secondary Schools and other societies meeting at the same time.

In May, 1922, Mr. Evans gives a summary of the report of the National Committee on Mathematical Requirements. Professor H. H. Rice, Worcester, discusses "Errors and Rounded Numbers," Professor Coolidge, "Geometrical Probability"; Professor L. P. Copeland, Wellesley, "Topics in History."

In December, 1922, Miss O. A. Kee, Boston, speaks on "Phases of Junior High School Mathematics," Professor Ralph Beatley, Harvard, on "The Civic Value of Mathematics," and Professor Lipka, M. I. T., on "Mathematics in Italy."

In May, 1923, novelty is lent to the program by "A Mathematical Trial" presented by members of the Mathematical Club of the Newton High School, and by a most graphic account of the early days of the telephone by Mr. Thomas A. Watson of Boston.

The above is by no means a complete list of papers and addresses, and the selection has been made not on the basis of merit or importance but with a view to indicating the range and some of the main tendencies of the programs. Particular apology may be made to the authors of numerous valuable and interesting papers on methods of teaching mathematical subjects, of which actual enumeration in this report seems impracticable.

Supplementing this chronological story, some further reference may be made to certain important phases of the work of the association:

Publications. A partial list of publications is as follows:

Second report of the association (presented at the third annual meeting held in Boston in 1906) with an account of the second annual meeting, November 19, 1905, and of the third spring meeting, April 15, 1906, followed by an alphabetical list of members.

Third report of the association presented at the fourth spring meeting, containing the constitution, the proceedings of the third annual meeting, the preliminary report of the Committee on Arithmetic, and the final report of the Committee on the Fundamental Propositions of Elementary Geometry, with an alphabetical list of members.

The geometry report was also printed by itself.

Second report of the Committee on the Teaching of Arithmetic, April, 1907.

Report of Committee on Sample Papers for College Entrance Examinations, December, 1907.

Loci, published in 1908, as noted under the year above.

Proceedings of the Association of Mathematical Teachers in New England, November, 1908.

Constitution and List of Members, December, 1911.

Minutes of the tenth spring meeting, April, 1912.

First report of the Committee on Mathematics of the Pre-High School Grades (without date).

Report of Committee on Secondary School Mathematics, April, 1916.

Report of Committee to Recommend a Suitable Program in Mathematics for the Junior High School, December, 1918.

External Relations. The association has frequently and advantageously cooperated with the Association of the Middle States and Maryland; as for example in the support of THE MATHEMATICS TEACHER. The attempt to cooperate on a broader geographical scale through the Federation of Teachers of Science and Mathematics in 1909 did not prove very successful. More recently the Association has had an interesting and important share in cooperation with the Mathematical Association of America in the organization of the National Committee

on the Teaching of Mathematics, the results of which can hardly fail to have far-reaching influence.

In Retrospect. Reviewing the record of these twenty years certain general aspects of the situation then and now are significant. When the association was established the relation between college and school men in mathematics, if not one of hostility, was certainly not one of cordial cooperation. In spite of pleasant personal contacts and friendships across the invisible boundary, there was a considerable survival of traditional exclusiveness on the part of the college men, with the corresponding combination of diffidence and criticism on the part of the school teachers. The college people were critical of the product of the schools; some of the school teachers were still more critical of the tests by which that product was measured. Through the work of the association and collateral agencies the relationship between the two groups has taken on an entirely different character. In the meetings and social gatherings of the association no boundary exists between school and college teachers, and in discussion there is free and profitable give and take. Criticism of entrance examinations is now of course aimed rather at the College Entrance Examination Board than at the individual college, but even then the responsibility attaches not to college teachers alone.

In the recent far-reaching work of the National Committee, college and school men and women have cooperated on equal terms. It is not too much to say that for this fortunate change in mathematical education this association deserves a substantial share of the credit.

W. F. Osgood,
Harvard University;

G. W. Evans,
High School. Charlestown, Mass.;

W. A. Francis, Exeter, N. H.;

C. A. Hobbs, Watertown, Mass.;

Roswell Parish, Brookline, Mass.;

H. W. Tyler,
Massachusetts Institute of Technology.

AN ANALYSIS OF THE TEACHING OF CANCELLATION IN ALGEBRAIC FRACTIONS

By AUGUST GROSSMAN
Ben Blewett Junior High School, St. Louis, Mo.

Every teacher of experience knows that a great many of his algebra pupils all the way from the first year in high school up to college continue with almost comical regularity to make strange mistakes in the subject of "cancellation" in fractions—mistakes that show clearly that the essence of the matter has escaped them. It is not necessary here to enumerate all the errors; suffice it to say that pupils are apt to cancel any two numbers or expressions that are the same, no matter in what context. For instance,

$$\frac{\cancel{8}x}{\cancel{8}y},\ \frac{\cancel{8}x+7}{\cancel{8}y+8},\ \frac{\cancel{8}x+\cancel{7}}{\cancel{8}+\cancel{7}},\ \frac{8(x+\cancel{7})}{9(5x+\cancel{7})},\ \frac{\cancel{8(x+7)}+11}{\cancel{8(x+7)}+2}$$

all look equally correct to them.

It is quite evident to the writer that the subject of "cancellation" has not been sufficiently analyzed and broken up in a simple way into its constituent elements. Certainly he has seen no such analysis in any text that has come under his observation. And it ought to be axiomatic that when in any subject under skillful and conscientious teachers pupils continue with regularity blindly to flounder, that subject is by reason of inherent difficulty not adapted to them at that particular stage of their development or that it *has not been adapted to them.* The writer is inclined to think the second hypothesis correct in this case.

The following method of analysis is here presented for what it is worth. The writer has found it to work reasonably well with his pupils. It is of course not intended that this is all to be swallowed by the pupil in one dose, or that it is to be given in an uninterrupted sequence; but the steps should be kept in mind.

The analysis is divided clearly into four steps:

1. Just what does "cancellation" mean?
2. What sort of cancellation do we employ in fractions?
3. How far does the effect of a *factor* extend?
4. "When can you cancel?"

I.

In order to get at the exact meaning of the word I begin with the every-day experience of the pupil—outside of his "algebra" experience, so to speak; that is, I begin with the *colloquial use* somewhat as follows:

Suppose a boarding house has a standing order at the grocer's for one dozen eggs each day. On Monday a letter arrives with the news that Uncle Joe, Aunt Mary and Cousin Jack will stop over on Tuesday on their way to Chicago. Mrs. Jones immediately phones to the grocer: "Add four more eggs to my order for this afternoon." The grocer, who was just making up the order, *adds four eggs* to the basket. No sooner has Mrs. Jones phoned, a telegram comes saying that the trip has been postponed. Mrs. Jones hurries to the phone and tells Mr. Grocer: "Please *cancel* that extra order of eggs." Mr. Grocer now goes to the basket and *takes out (subtracts) four eggs.* The order which had formerly stood as $12 + 4$ now stands as $12 + \cancel{4}$.

What did the "cancellation" of "$+ 4$" mean? Clearly it meant *taking away (subtracting) 4.*

Some time after this, two of her boarders suddenly leaving town, Mrs. Jones again phoned the grocer: "I am losing two of my boarders; send up four eggs less after this." Mr. Grocer, who had already made up the order, immediately *takes 4 eggs* from the basket, *i. e.,* he *subtracts 4 eggs.* Five minutes later he phones Mrs. Jones: "There are two travelling salesmen here and they can't get accommodations at the hotel. Can you take them for a week?" Mrs. Jones is glad to take them, thanks the grocer, and winds up by saying: "By the way, never mind what I said a few minutes ago about bringing less eggs; leave the order just as it's always been." In other words, she *cancelled* her phone order of "$- 4$ eggs." The order which formerly stood as $12 - 4$ now stands as $12 - \cancel{4}$ (*i. e.,* 12). The cancellation of "$- 4$" evidently means the *addition* of $+ 4$.

In the same way if a person has ordered x gallons of milk, and then doubles the order, it becomes $2x$ gallons. Cancelling the message "double the order" leaves the order $\cancel{2}x$ (*i. e..* x) gallons which amounts to a *division* by 4.

Similarly, if we say $x/2$ or $x \div 2$, the cancellation of "2" leaves these quantities as x, which amounts to a *multiplication* by 2.

Reviewing, then, we find a cancellation of an addend means subtraction; of a subtrahend, addition; of a multiplier, division; and of a dividend, multiplication. In other words, when you "cancel" you always perform an operation which is the reverse of the one which was effected by the quantity cancelled.

It might be well to state that many such illustrations should be given, and that pupils might be encouraged to give them, or that they might be assigned the task of bringing them in as a regular lesson. Parenthetically it may be aded that a half hour's or an hour's time spent by the teacher in picking out the best (most striking) illustrations to make clear a new concept is a time-investment that pays large dividends.

It is, of course, desirable at this point, to take many *simple* expressions in which cancellations have taken place, and to tell what has been the effect of the cancellation.

II.

We are now ready to *recall* to the pupil (it is not necessary to teach it anew) what sort of cancellation we employ in fractions.

We must note first that in all our cancellation we are expected *not* to *change* the *value* of the *fraction.* Most pupils will without much trouble see that $\frac{3(2)}{5(2)}$ is the same as $\frac{3}{5}$; that $\frac{3+2}{5+2}$ is not the same as $\frac{3}{5}$; that $\frac{3-2}{5-2}$ is not the same; and that $\frac{6}{10}$ is the same as $\frac{6 \div 2}{10 \div 2}$. In other words, the fundamental concept in cancellation in fractions, which is so obvious to one accustomed to handling fractions, but which is not so obvious to the beginner, must be made *absolutely clear;* namely, that the numerator and denominator may at the same time be either multiplied or divided by the same number without thereby altering the value of the fraction, but that the value of the fraction *does change* if the same number be at the same time either added to both numerator and denominator, or subtracted from

them. In other words, we may cancel the same *factor* (multiplier) from both numerator and denominator, for we thereby divide both by the same number. (Call attention again to the fact that the cancellation of a multiplier means division.)

To a teacher who, through habit, has learned automatically to do all this, it seems so obvious as to need no emphasis. The fact is, that the idea is so fundamental, that emphasis is just what it needs. The writer has had the pleasant experience this summer of dipping into an entirely new subject under a very brilliant young instructor. To the B. Y. I. everything was so obvious that at any given time not more than half the class understood him, and that half the time nobody understood him.

Get across, then, the idea that *the value of a fraction is not changed when both numerator and denominator are both either multiplied or divided by the same number, but that the value is changed if the same number be at the same either added to both numerator and denominator or subtracted from them.* Learn this statement word for word, if necessary. Rote learning of a rule is at all times to be condemned when it is merely rote learning, but not if it is empolyed *after* the idea underlying the rule is clearly understood. It is then a time and labor saver.

III.

HOW FAR DOES THE EFFECT OF A MULTIPLIER EXTEND?

This is the real source of many of the mistakes in cancelling in fractions. A pupil may understand all that has gone before and still cancel wrongly through not understanding how far the division effect of the cancellation of a factor extends.

The essence of the idea is that it extends as far as the multiplying effect of the factor itself extends. And here it is necessary to show the pupil this matter by *actual concrete cases at first,* and *then* with generalized expressions.

If in the expression 3(4) we cancel 3, we have $\cancel{3}(4)$, *i. e.,* 4; we have divided by 3. The expression 3(4) is the same as 4(3); *i. e.,* the order of the factors is immaterial; as long as either factor is cancelled, the other factor remains. In the expression 3(4)(5) the force of the 3 as a multiplier extends to the whole expression *once* and it does not matter where the 3 is placed; *i. e.,* 3(4)(5), (4)(3)(5), 4(5)(3) are identical. In the expres-

sion $3(4)(5)+7$ the force of the 3 as a multiplier extends as far as the "+" sign and no farther. Here it might be well actually to show that $4(5)+7=20+7$, and that $3(4)(5)+7=60+7$. (This recalls the old rules which I ask my pupils the first few days of the term to recite in concert: "Times and division signs connect, plus and minus signs separate." It is a re-emphasis on what we mean by a "term.")

Just as in $3(4)(5)+7$ the force of any of the factors, 3, 4, or 5 extends only as far as the + (or —) sign, *i. e.*, only to the term which contain it, so in $x(a)(y)+7$, the force of either x, a, or y extends only to the + (or —) sign.

A few illustrations and questions are now desirable:

In the expression $3(\text{anything})y+7$ there are three factors; the force of any of them extends only as far as the + sign.

In $3(a+b)y+7$ the effect of either of the three factors extends only as far as the + sign.

In $7a+8b$, 8 multiplies b only.

In $7(a+b)c-3m$ what does $(a+b)$ multiply?

In $7(x+y)3(c+m)$ what does $(x+y)$ multiply?

In $6(x+y)(c+b)-3(m+5)$ what does 3 multiply?

IV.

"WHEN CAN YOU CANCEL?"

My procedure in answering this question is about as follows:

(a) Recall to the pupil now that we must divide both numerator and denominator by the same number; *i. e.*, if we cancel, we must cancel the same factor.

(b) Put down on the board a dozen or more expressions of varying degrees of complexity:

$$\text{(A) } \frac{3x}{3y},\quad \text{(B) } \frac{3x+y}{3a+b},\quad \text{(C) } \frac{3(a+b)}{3(x+y)},\quad \text{(D) } \frac{3(a+b)-5}{3(a+b)+7},$$

$$\text{(E) } \frac{7(x+y)(a-b)}{3(x+y)(a+b)},\quad \text{(F) } \frac{8(x+y)+5}{3(x+y)},\quad \text{(G) } \frac{7(x+y)5}{3(x+y)b},$$

$$\text{(H) } \frac{5(c+y)(a-b)}{3(c+y)(2a-b)},\qquad \text{(I) } \frac{3(x+y)2(a+b)}{7(x+y)2(a+b)}.$$

(c) Now begin to cancel factors in these expressions and see whether the cancellation really does divide the *whole* numerator and the *whole* denominator.

By having all these before you, you see at once that such cancellation is possible in A, C, E, G, H, and I; that is, when both numerator and denominator are in the form of an *uninterrupted succession of factors*. This last sentence is the crux of the whole matter.

Stated in this way the matter looks very simple, but I believe it is necessary to go through some such analysis as the above to give it axiomatic vividness.

THE QUESTION OF METHOD IN SUBTRACTION

By WILLIAM F. ROANTREE
New York Training School for Teachers

It is the purpose of this article to set forth evidence to show the superiority of the so-called Austrian method of subtraction over what is sometimes called the Italian method. To define the methods named precisely it is necessary to describe them and to state the principle of each.

$$\begin{array}{r} 524 \\ 238 \\ \hline \end{array}$$

In working the example by the Austrian method one thinks: 8 and 6 (write 6) are 14, 4 (carrying 1 from the sum 14) and 8 (write 8) are 12, 3 and 2 (write 2) are 5. The principle employed is: *Increasing the subtrahend by the remainder gives the minuend.*

In working the same example by the Italian method one thinks: 8 from 14, 6; 3 from 11, 8; 2 from 4, 2. The principle is: *Taking one from any order of a number and adding ten to the next order to the right does not change the value of the number.*

A few years ago the New York City course of study in arithmetic was in the hands of a committee for revision. One of the points under discussion was the proposition to adopt the Austrian method as the uniform method to be used in all the schools. It was hoped by this means to protect the children from the confusion which must result when they are required to change from one method to another upon being transferred to a new school. In cases where the children concerned were in a grade after subtraction had been started and before it had been finished, such transfers very frequently resulted in interference with partly established habits. At that time the writer had printed a supply of the appended test in subtraction, and on the 22nd of that month gave the test to 2B (latter half of second year) classes in five New York City schools; two of these schools were located only a few blocks apart on the west side below 59th Street, one using the one method and the other using the other method, and the remaining three schools were located east of

THE TEST

Subtract:

815 318	800 268	931 388	697 298	723 365	483 73
766 86	682 94	825 617	550 441	740 493	810 502
961 463	968 289	904 648	844 65	530 174	621 489
467 99	552 93	943 154	211 197	691 484	375 71
980 879	970 96	210 100	379 106	803 97	322 176
751 285	425 296	564 304	824 149	433 196	277 62

Central Park above 86th Street. At the time, the west side schools were attended by about the same class of children, also in the case of the east side schools the social and racial conditions were fairly uniform. Of the 346 children examined 177 had been taught the Italian method and 169 the Austrian method. Five minutes was allowed for the work, and only a very few children finished all the examples.

The results showed a fairly normal distribution so far as *attempts* were concerned, but the distribution of *rights* showed that a considerable number of children had not yet gotten the "hang" of the work; consequently there was a rather large group that split off from the rest, and were plainly not ready to take such a test. The results reveal the following significant facts:

1. In the east side classes the Austrian method showed an accuracy score of 77 per cent (number of *rights* divided by number of *attempts*), and the Italian method an accuracy score of 67 per cent. Along with this substantial superiority in accuracy,

the Austrian method showed a marked superiority in speed. The median scores of *attempts* were, for the Italian method 8 examples (24 combinations), and for the Austrian method 12 examples (36 combinations). It should be explained here that these speed scores are based not on the total distributions, but upon the *attempts* scores of those who in each group were above the median score in accuracy. These measures seem more significant for the reason that they are not affected by the work of those children who were unable to do anything with the test, many of whom, however, rolled up large *attempts* scores by writing at random whatever seemed to tickle their imaginations.

2. In the west side classes the median accuracy scores were, for the Italian method, 45 per cent, and for the Austrian method, 50 per cent; the median speed scores for those who were above median in accuracy were, for the Italian method, 7 examples (21 combinations), for the Austrian method, 8 examples (24 combinations). In this comparison the Austrian method was at a disadvantage. In the school using the Austrian method the change to this method had been made about March 1st when a new head of department assumed her duties. The children examined had in the preceding grade learned the subtraction combinations in their *take-away* sense; only about six weeks intervened between the change of method and the test. The department head who was responsible for the mid-stream swap in methods told me that her teachers were entirely unfamiliar with the new method when the change was proposed. It would seem that the comparative success of the method was due to the zeal with which it was launched; for, as a rule, it is unwise to change from one habit to another unless the former habit is so bad that it gives no promise of functioning usefully.

Accuracy Score in per cents	Proportion of Papers Making the Score Italian Method	Austrian Method
90 to 100	15 per cent	26 per cent
80 to 89	20 " "	18 " "
70 to 79	10 " "	12 " "
60 to 69	10 " "	9 " "
50 to 59	8 " "	6 " "
0 to 49	37 " "	29 " "

The above table, based upon the east side scores, shows in detail the superiority of the Austrian papers with respect to

accuracy. The table shows that 15 per cent of the Italian papers and 26 per cent of the Austrian papers achieved an accuracy of from 90 to 100 per cent, etc.

Taken at their face value the facts above cited would indicate that the Austrian method is superior to the Italian method; and the difference was sufficiently pronounced to suggest to the writer the desirability of making a further study of the question. On the chance that the committee previously mentioned would decide in favor of the Austrian method, the writer resolved to give the same test the last week of every term in the Model School of the New York Training School for Teachers and in another nearby school until he had secured a number of comparisons of the methods in question in which the *teacher variable*

TABLE A

Scores No. of Com's.	Miss G's Classes Italian		Austrian		Miss M's Classes Italian		Austrian		Miss C's Classes Italian		Austrian	
	Att.	Rt.	Att.	Rt.	Att.	Rt.	Att.	Rt.	Att.	Rt.	Att.	Rt.
0 or 1	5	6		1		2		2				
2—3	3	8		1	1	5		4				
4—5	6	5		1		5		1	1	2		
6—7	3	4	1	1	7	5		2	1	5		3
8—9	4	11	1	3	5	6	2	3	3	5	1	1
10—11	12	2	4	1	7	7	1	3	2	6		4
12—13	4	3	1	2	12	5	11	6	8	4	4	7
14—15	2	1	1	3	1	1	4	4	4	5	6	5
16—17	2	2		4	2	1	2	1	2	2	3	2
18—19	1	1	6	5	1	1	7	4	3	1	5	1
20—21	1		6	2	2	1	2	2	4	3	8	9
22—23			4	4	2	1	3	4	5	2	5	2
24—25			3	3							4	2
26—27			2				2		2	1	4	7
28—29			3	1					1	1	1	1
30—31				1					1		2	
32—33			1								1	
34—25			2	3			1	1				1
36—37			2	1			3	1			1	
38—39												
40—41								1				
42—43							1					
No. in Class	43		37		40		39		37		45	
Median Scores	10	7.5	21.5	19	12	9	17	14	15	12	21	19
Accuracy	.75		.88		.75		.82		.80		.90	

would be eliminated. It was hoped that such comparisons could be obtained in each grade from 2B to 4B, but in many instances the teachers whose classes were tested under the old method were assigned to different grades or transferred to other schools, thus making it impossible to complete the comparisons. It was possible, however, to get enough data on the 2B grade to convince the writer that the Austrian method gives the best results in that grade. In order to make the test adaptable to all grades the time was reduced to two minutes; there was no case in which a child worked all the examples in two minutes.

Table A shows three comparisons of the methods in question, based upon the results achieved by classes in the 2B grade conducted by three teachers, Miss Gage and Miss Morey (since retired) of the Model School and Miss Cannon of P. S. No. 10, Manhattan.

The per cents of accuracy were obtained by dividing the median of *rights* by the median of *attempts*.

When the three sets of distributions in Table A are combined into one set of distributions, the median scores for the Italian method are found to be 12 *attempts* and 9 *rights*, and for the Austrian method 20 *attempts* and 17 *rights*. The accuracy for the Italian method is 75 per cent, and for the Austrian method 85 per cent. As measured by the *attempts* scores the Austrian method shows 67 per cent greater speed than the Italian method; if the *rights* scores are used the speed superiority is 89 per cent.

The reader has doubtless noted that the results have been worked out on the basis of the simple combination. This seemed better considering the fact that the children in 2B are so young. If the example were the unit in scoring, a child might get two-thirds of his combinations right and yet get a zero score. Table B shows the results scored by using the example as the unit. All the classes are combined, giving 120 Italian papers and 121 Austrian papers. The accuracy scores drop about 15 points below what they are by the other method of scoring.

The unwisdom of trying to change to the Austrian method after the children have learned the combinations in their *take-away* sense is illustrated by the following: Miss Morey taught second year classes three years in succession; in June, the first year. the Italian method yielded median scores of 4 examples attempted and 2.2 examples right; the next June the Austrian

TABLE B

No. of Examples	Italian Att.	Italian Rt.	Austrian Att.	Austrian Rt.
0	5	25		10
1	8	25		11
2	15	20	3	13
3	30	20	8	16
4	28	13	19	13
5	11	5	13	13
6	7	4	23	16
7	12	5	23	10
8	1	2	12	9
9	2	1	6	2
10	1		3	2
11			4	3
12			6	3
13				
14			1	
No. of Scores	120		121	
Median	4.1	2.5	6.9	4.8
Accuracy	.61		.70	

method, following the Italian method in the preceding fall term, yielded median scores of 4.75 examples attempted and 2.25 examples right. The following June the Austrian method, used from the beginning, yielded median scores of 6 examples attempted and 4 examples right.

The only comparisons obtained in grades above 2B were one comparison for 3A, one for 3B, and one for 4B. The results are shown in Table C. For the first two classes the combination was the unit in scoring, in the last class the example was the unit. In the 3A comparison the Italian scores are ahead, but Austrian scores show a higher degree of accuracy—93 per cent as against 87.5 per cent. On the whole, these comparisons reveal no significant differences.

TABLE C

Class	Italian Att.	Italian Rt.	Austrian Att.	Austrian Rt.
3A	24	21	21	19.5
3B	25.5	24.5	26.6	25.7
4B	12.6	11.1	13.6	12.6

What are the conclusions to be drawn from the data set forth in these tables, and how reliable are these conclusions?

First, the superiority of the Austrian method in the work of the second year indicates that this method is more easily learned than is the Italian method.

Second, it probably makes little, if any, difference which method one uses, *assuming that the method used has been thoroughly mechanized.*

Third, the Austrian method is preferable to the Italian method for the reason that, if properly taught, it requires less time and effort to master it.

My second conclusion agrees with the results of other experiments, but the experimenters have made the mistake of assuming that there is no preference as between these methods because tests given in grades *above the third year* reveal no difference.

The teachers whose classes were examined were well known to the writer as being experienced, and fully capable of carrying out their part in the experiment. The time devoted to subtraction when the Austrian method was being taught was never in excess of the time given when the Italian method was used; one teacher reported a decrease of time.

DISCUSSION

Some Time-Saving Methods in Teaching Graphing. The purpose of this paper is to show how teachers may save time and effort in grading exercises in the graphing of linear and quadratic equations.

If in equations of the type

$$ax + by = c,$$

We assume for, say, x an arithmetical series of values whose common difference is b, then the corresponding series for y will have a common difference a. Common differences other than a or b may be used, but if one series is arithmetical the other is also. If c is some integral multiple of a and b, then fractions are avoided under the first suggestion above.

If teachers do not "put the pupils wise" to the whole secret, they could at least show them how to obtain the first series. Then, if the corresponding series is not an arithmetical progression, the teacher will know what and where the trouble is.

It is also interesting even to first year pupils to find the equation for two given corresponding arithmetical series, showing that every pair of such coordinates will check in the derived equation. To illustrate, assume the two arithmetical series:

$x =$	9	6	3	0	-3	-6	-9
$y =$	-10	-8	-6	-4	-2	0	2

If we assume the form,

$$x + sy = t,$$

to which $ax + by = c$ reduces if we divide by a; and if we substitute, say, the first two pairs of coordinates above in this equation, $x + sy = t$,

we have the equations,

$$9 - 10s = t$$
$$6 - 8s = t,$$

whence $\quad s = 3/2$ and $t = -6$.

Then $x + sy = t$ becomes

$$x + 3/2 y = -6,$$

or $\quad 2x + 3y = -12.$

Note that any pair of the series assumed above satisfies this equation.

Now, as to the quadratic, $f(x) = ax^2 + bx + c$· Consider the special function,

$$f(x) = x^2 + x - 6.$$

Assuming an arithmetical series for x, we may have

$x =$	-4	-3	-2	-1	0	1	2	3
$f(x) =$	6	0	-4	-6	-6	-4	0	6
Differences =		6	4	2	0	-2	-4	-6

where the third or bottom row is also an arithmetical series obtained by subtracting each term of the middle row (except the first term) from the term immediately preceding it.

So, if a class is given an arithmetical series for the x row, then the second, or $f(x)$ row, should, when the terms are subtracted as above, give the teacher an arithmetical series as a check.

It might be noted that when the quadratic is of the form.

$$f(x) = x^2 + bx + c,$$

and if the x-row is consecutive numbers, then the third row has the number 2 for a common difference.

An analogous scheme "works" for the cubic equation, but it requires four rows to include an arithmetical progression, and is not especially recommended.

If the quadratic is of the form

$$f(x) = ax^2 + bx + c,$$

the common difference is always $2ad^2$, where d is the common difference in our first row. This may be shown by substituting $x - d$, $x - 2d$, $x - 3d$, etc., for x in the function $ax^2 + bx + c$, and subtracting consecutive results, thus obtaining an arithmetical progression whose common difference is $2ad^2$.

Also, classes enjoy finding the quadratic function corresponding to a three-line arrangement, as below.

$x =$	5	3	1	-1	-3	-5	-7
$f(x) =$	20	12	7	5	6	10	17
Differences =		8	5	2	-1	-4	-7

where the row of "Differences" is any arithmetical progression assumed before the other two rows are obtained. Then the $f(x)$ row is started with any convenient number (20 in this case), and is completed by filling in such numbers that the differences between the consecutive terms will give the already assumed third row. Then the x row can be any assigned arithmetical progression.

Now, to obtain the corresponding function, substitute in $f(x) = ax^2 + bx + c$, say, the second, third, and fourth columns of the coordinates above, and we have the equations,

$$12 = 9a + 3b + c$$
$$7 = a + b + c$$
$$5 = a - b + c,$$

whence $a = 3/8$, b = 1. and $c = 45/8$.

Substituting these values of a, b, and c in the function,

$$f(x) = ax^2 + bx + c,$$

we have $f(x) = 3/8 x^2 + x + 45/8$,

in which each of the other columns happens to check. It is best, however, not to multiply the function by the common denominator before checking.

M. H. Pearson.

Sidney Lanier High School,
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RECENT ARTICLES

American Mathematics During Three-Quarters of a Century, Science, for January 4, 1924; an address by Professor G. A. Miller, of the University of Illinois, before Section A, American Association for the Advancement of Science, at Cincinnati, Ohio, December, 1923.

The Practical Value of Pure Science, Science for January 4, 1924; an address by Dr. Robert Andrew Millikan, of the Norman Bridge Laboratory of Physics, Pasadena, Cal., when he was presented with the Edison Medal, October 4.

The Influence of Astronomy upon Modern Thought, Popular Astronomy for January, 1924; an address by Dr. Heber D. Curtis, at the dedication of the Irving Porter Church Memorial Telescope at Cornell University, June 15, 1923.

The last two articles named are suggestive and inspiring to the teacher of mathematics.

Alfred Davis.

NEWS NOTES

THE Philadelphia Section of the Association of Teachers of Mathematics of the Middle States and Maryland held the following program in Philadelphia, October 25, 1923: "High School Class Instruction in the Use of the Slide Rule," Mr. Harry Ginsburgh, Department of Mathematics, Central High School; "The Problems of the Solar Eclipse of 1923" (illustrated), Dr. John A. Miller, head of the Department of Mathematics, Swarthmore College.

THE Association of Teachers of Mathematics in New England met on Saturday, January 19, 1924. The program consisted of short talks on mathematics by six past presidents, including: Charles A. Hobbs, Archibald V. Galbraith, Julian L. Coolidge, Harry B. Marsh, William R. Ransom, Walter F. Downey. H. D. Gaylord is secretary of the association.

THE New York Section of the Association of Mathematics Teachers of the Middle States and Maryland held the mid-winter meeting, January 18, 1924, at the Lincoln School of Teachers College, 425 West 123rd Street, New York City. The following program followed a dinner:

Welcome—Dr. Otis W. Caldwell, Director of The Lincoln School
Business Meeting (Brief)....................7:30 P. M.
A Better Use of Tests,
Professor W. D. Reeve, University of Minnesota
Discussion, Maurice Crosby, Bronxville, N. Y.
How to Classify Pupils for Instruction in Algebra,
W. E. Breckenridge, Stuyvesant High School, New York
Discussion, Winona M. Perry, Providence, R. I.
Should Statistics Be Offered as an Elective Subject in the Senior High School?
Professor Godfrey Thompson, University of Durham, England
Discussion, Vera Sanford, The Lincoln School.

A prize of $1,000 has just been awarded to Leonard Eugene Dickson, Professor of Mathematics in the University of Chicago, at the seventy-fifth meeting of the seventy-fifth meeting of the

American Association for the Advancement of Science, held in Cincinnati. The prize awarded Professor Dickson was offered by a member of the association in Cincinnati for the most important piece of work contributed at this meeting. Professor Dickson's contribution was on "The Arithmetics of Higher Number Systems." Professor Eliakim Hastings Moore, head of the department of Mathematics at the University, says of this work that "it opens a large and very important new field in the theory of numbers."

Professor Dickson received his S.B. and A.M. degrees from the University of Texas in 1893 and 1894 respectively, and his Ph.D. degree from the University of Chicago in 1896. He also studied at the Universities of Paris and Leipzig from 1896 to 1897. He is a member of the National Academy of Sciences and of the National Research Council. He has received a number of medals and prizes from learned societies, and is the author of *Algebras and Their Arithmetics*, published by the University of Chicago Press.

ALFRED DAVIS.

NEW BOOKS

Mathematics. By DAVID EUGENE SMITH. Vol. 36 of *Our Debt to Greece and Rome.* Marshall Jones Company, Boston, 1923; pp. 175 + x.

If we were asked to itemize our mathematical debt to Greece and Rome, I fancy we would all write down demonstrative geometry at once, but that then we would hesitate, wondering what other specific things we might add. After reading this little volume, we are in the opposite predicament: is there anything that we study in mathematics today that does not have its root in the mathematics of one or other of those two countries? We are brought to agree with the initial thesis of Sir Thomas Heath's introduction:

"The history of the beginnings of mathematics in the sense in which we understand the term is the history of mathematics in Greece; for it was the Greeks who first conceived the notion of mathematics as a science in and for itself, and it was they who established mathematics as a logical system based upon a few elementary principles, principles which they were the first to lay down and which remain substantially unshaken to the present day."

The book itself is not to be read at a single sitting. A "Preliminary Survey" is followed by two sections that make up the major part of the book: "Contribution in Detail" and the "Influence of These Contributions." After you have read the first chapter, you may wish to take liberties with the table of contents by following a single topic such as algebra through both of the main divisions of the work, then studying another topic, etc. Or you may prefer the logical order that the author used. It depends on whether you are more interested in the subject as a whole or in its component parts taken separately. Whichever way you use, you are certain to wish that the book might have been longer so that you might have learned more about each subject. You make note of certain topics as *number systems, finger reckoning,* and the *abacus* for reference work with high school classes. For yourself, you resolve that instead of unwittingly acknowledging your debt to the mathematics of Greece and Rome, by using terms of Greek or Latin derivation thought-

lessly, you will be more conscious of that debt and will endeavor to discharge that obligation by making your pupils see the "sweet reasonableness" of these terms when one knows their connotation to the people who first applied them to mathematics.

Finally, to those of us who have had the good fortune to be in Dr. Smith's classes in the *History of Mathematics,* this book will bring many recollections of those days. We will supplement it with the obiter dicta that were of necessity crowded from these few pages. For others who have been less fortunate as yet, the book cannot replace these experiences , but it certainly offers more than the proverbial "half a loaf."

VERA SANFORD.

The Lincoln School of Teachers College.

Supervised Study in Mathematics and Science. By S. CLAYTON SUMNER. The Macmillan Company, 1922. Pp. 241.

Little of what we teach today will be remembered a decade later; only a small modicum can be recalled at the end of the current year. Is teaching, then, futile? Not if it results in the development of certain attitudes of mind and habits toward society, toward work, toward life. Not if pupils learn how to investigate, how to think, or more commonly, *how to study.* The keener recognition of the possibilities and importance of teaching how to study a subject as well as teaching the subject led to the supervised study movement. What was supervised study? Usually a lengthened class period, divided into a "recitation" period and a "study" period, with the elimination of home work. At 9 o'clock the geometry class was called; it recited until 9:30, when it suddenly engaged in a study of a preparation for the next lesson. Supervised study was an administrative change; few of those who administered conceived it as *purposeful learning, learning under skillful guidance,* giving an opportunity for practice in problem solving with just enough aid to assure success without removing all of the obstacles.

More recently the need for a fundamental interpretation of the theory and a detailed description of the operation of supervised study became apparent. Dewey's *How We Think,* McMurray's *How to Study,* and the practice of artist-teachers had to be translated into the pedagogy of the school subjects. Un-

der the direction of Professor Alfred Hall-Quest this work has been undertaken. This volume, *Supervised Study in Mathematics and Science,* was prepared to describe not its theory, but its practice. Numerous excellent suggestions of how to teach various topics in algebra and geometry are included. The treatment given to the sciences seems far less adequate.

The author proposes a three-fold division of the class period, as follows: The Review, 15 minutes; the Assignment, 20 minutes, and the Silent Study, 25 minutes. "The review takes the place of the old recitation. The assignment is always the most important part of the period. The study section is the part devoted by the pupil to the study of the advance lesson, under the direct and sympathetic supervision of the teacher." The three-fold *assignment* scheme includes a minimum average, and maximum assignment, thereby providing for different ability groups.

The undersigned is in entire sympathy with the purposes of supervised study, and in almost entire disagreement with its conventional procedures. In the first place, it substitutes a massage treatment of the curriculum for a seriously needed surgical operation. Through mere devices it attempts to correct or aleviate the short comings of improperly devised teaching materials. A curriculum in mathematics that *challenges* the pupil; that guides by means of adroit questions and suggestions; that moves, not always in the straight lines of the logician, but in the somewhat circuitous path of the learner; that is made up of numerous and varied activities such as tables to be completed, matching exercises, statements to be judged, nuts to be cracked, short cuts for saving time, goals to be reached; that is personal, human, practical; that is not above or below the ability of the class—such a curiculum generates its own drives for thinking, draws on genuine purposes, and leads to "confidence in problem solving." The threefold division of the class hour becomes formal, deadening; reliance upon the use of historical notes, preachments about the importance of the subject in the racial heritage and the obligations of pupils to study dutifully, become unnecessary. Should not we who appreciate the importance of effective study habits take an uncompromising stand for the construction of curriculum materials in accordance with the psychology of the learner?

With adequately designed teaching materials the need for an *imposed* technique to be applied to *assignments* will disappear.

JOHN R. CLARK.

The Lincoln School of Teachers College.

A Course in Arithmetic for Teachers and Teacher-Training Classes. By JAMES ROBERT OVERMAN. Lyons and Carnahan. 1923. pp. 376.

Students entering normal schools to prepare themselves for teaching in the elementary schools are as a general rule in need of a review of the elements of percentage and its simplest applications. Indeed many even need a review of common fractions, decimals, and the mensuration of the plane figures. Instructors in normal schools conducting classes in the teaching of arithmetic are obliged therefore to devote a part of their time to teaching and reviewing topics that should be familiar to the students before entering normal schools. They must also help their students to equip themselves with a knowledge of some topics in arithmetic beyond that necessary just to be able to teach the minimum essentials. All this must be done in addition to the study of the methods and ways of teaching every topic in arithmetic in the elementary school. The methods course is extensive in itself but in this course considerable time must be given to reviewing and extending subject matter. There has been a demand for a text containing a sufficient amount of both types of work, offering the opportunity of studying methods of teaching in connection with a review of old material and a study of new subject matter. *A Course in Arithmetic for Teachers and Teacher-Training Classes* is just such a text, entirely free from elaborate discussions of extraneous topics like formal discipline, general motivation, etc., now treated in texts on various phases of general education.

This book contains seventeen chapters. Seven are devoted to the teaching of the fundamental operations in whole numbers, common fractions, and decimals; single chapters cover each of the subjects, the fundamentals of percentage, denominate numbers, geometry of position, geometry of form, geometry of size (common mensuration, square root, and angular measurement), graphical representation, and the teaching of the fundamental operations in algebra; and four chapters treat the arithmetic of

the home, savings and investments, civic life, and business. Included in the discussions and at the close of chapters are over two hundred review questions and practice exercises in computation and over three hundred practical problems advanced enough to give normal school students the kind of practice and study that they need.

It is encouraging to have access to a text of this kind. It may not be wholly satisfactory because of certain omissions, occasional brevity, or unnecessary treatment of a few superfluous topics, but in the main it is a vast improvement over the general run of texts on this subject in meeting the real needs of students preparing themselves to teach arithmetic.

E. C. HINKLE.

Chicago Normal School.

Vector Analysis. By C. RUNGE, Professor of Mathematics at the University of Gottingen; translated by H. Levy, Assistant Professor of Mathematics at the Imperial College of Science, London; 226 pp. E. P. Dutton and Company, New York, 1919. Price, $3.50.

The Chapter headings are:

Vectors and Vectorial Areas.

The Differentiation and Integration of Vectors and Vectorial Areas.

Tensors.

From the author's preface:

"This, the first volume, contains the vectorial analysis of three dimensions. In the second volume, that of four and more dimensions, playing an important part in the theory of relativity, will be treated."

ALFRED DAVIS.

Economics of the Family. By C. W. TABER and RUTH A. WARDALL. J. B. Lippincott Company, Philadelphia. Pp. 224; 1923.

This book suits admirably the purpose for which it was written, namely, a high school text-book for courses in home and personal economics. It is also of great value to teachers of mathematics who give instruction in business practice and per-

sonal accounts. The writer finds the discussion of budgets, household accounts, the family and the bank and business projects for boys and girls an extremely suggestive and helpful source of information for the teaching of these topics in junior high school mathematics.

Economics of the Household. By Benjamin R. Andrews. The Macmillan Company. Pp. 623. 1923.

This book treats the general administrative and financial background of the private family household. It is a very complete and scholarly discussion intended as a college text.

Essentials of Plane and Solid Geometry. By David Eugene Smith. Ginn and Company. Pp. 504. 1923.

Problem Arithmetic. An Inductive Drill Book, by Harry Brooks. Little, Brown and Company, Boston. Pp. 331. 1923.

Intended for grades seven and eight and for junior high schools.

Modern Business Mathematics. By George H. Van Tuyl. American Book Company. Pp. 307. 1923.

Practice Exercises for Accuracy and Speed in the Fundamentals of Arithmetic. By Raleigh Schorling and John R. Clark. The Gazette Press, Yonkers, N. Y.

A 32-page booklet containing 46 timed tests, with provision for individual records of progress.